POSITIVITY OF DIRECT IMAGES OF FIBERWISE RICCI-FLAT METRICS ON CALABI-YAU FIBRATIONS

YOUNG-JUN CHOI

Abstract. Let X be a Kähler manifold which is fibered over a complex manifold Y such that every fiber is a Calabi-Yau manifold. Let ω be a fixed Kähler form on X. By Calabi and Yau's theorem, there exists a unique Ricci-flat Kähler form $\rho|_{X_y}$ for each fiber, which is cohomologous to $\omega|_{X_y}$. This family of Ricci-flat Kähler forms $\rho|_{X_y}$ induces a smooth $(1,1)$-form ρ on X with a normalization condition. In this paper, we prove that the direct image of ρ^{n+1} is positive on the base Y and that there exists a lower bound of ρ which is given by the Green kernel on each fiber and the Weil-Petersson metric on Y. We also discuss several byproducts, among them the local triviality of families of Calabi-Yau manifolds.

Contents

1. Introduction

Let $p : X \to Y$ be a proper surjective holomorphic mapping between complex manifolds X and Y whose differential has maximal rank everywhere such that every fiber $X_y := p^{-1}(y)$ is a compact Kähler manifold. This is called a *smooth family of compact Kähler manifolds* or a *compact Kähler fibration*. If every fiber X_y is a Calabi-Yau manifold, i.e., a compact Kähler manifold whose canonical line bundle K_{X_y} is trivial, then the family is called a *smooth family of Calabi-Yau manifolds* or a *Calabi-Yau fibration*.

2010 *Mathematics Subject Classification.* 32Q25, 32Q20, 32G05, 32W20.
Key words and phrases. Calabi-Yau manifold, Ricci-flat metric, Kähler-Einstein metric, a family of Calabi-Yau manifolds, variation.
This work is supported by the ERC grant ALKAGE at Université Grenoble-Alpes.

If (X,ω) is a Kähler mainfiold, then a celebrated theorem due to Calabi and Yau implies that on each fiber X_y, there exists a unique Ricci-flat metric $\omega_{KE,y}$ in the cohomology class $[\omega|_{X_y}]$. This family of Ricci-flat metrics induces a fiberwise Ricci-flat metric on the total space X.

The main theorem of this paper is the following:

Theorem 1.1. *Let $p : X \to Y$ be a smooth family of Calabi-Yau manifolds. Suppose that X is a Kähler manifold equipped with a Kähler form ω. Let $\omega_{KE,y}$ be the unique Ricci-flat form in the cohomology class $[\omega|_{X_y}]$. Then there exists a unique smooth function $\varphi \in C^\infty(X)$ which satisfies the following properties:*

(i) $\int_{X_y} \varphi(\omega_{KE,y})^n = 0$ *for every* $y \in Y$,
(ii) $\omega + dd^c\varphi|_{X_y}$ *is a Ricci-flat Kähler form on* X_y *for every* $y \in Y$ *and*
(iii) $p_*\left(\omega + dd^c\varphi\right)^{n+1}$ *is a positive* $(1,1)$*-form on* Y.

Here d^c means the real operator defined by

$$d^c = \frac{\sqrt{-1}}{2}\left(\partial - \bar\partial\right).$$

Then we have $dd^c = \sqrt{-1}\partial\bar\partial$. We call the $(1,1)$-form $\rho := \omega + dd^c\varphi$ which satisfies the property (ii) a *fiberwise Ricci-flat metric* or a *fiberwise Ricci-flat Kähler form* on a Calabi-Yau fibration $p : X \to Y$. Note that a real $(1,1)$-form on X satisfying (ii) is not uniquely determined. With the normalization condition (i), the fiberwise Ricci-flat metric is uniquely determined. From now on, the fiberwise Ricci-flat metric on a Calabi-Yau fibration means the real $(1,1)$-form which satisfies (i) and (ii). It is remarkable to note the following:

- Theorem 1.1 basically deals with a smooth family of polarized Calabi-Yau manifolds in the sense of deformation theory.
- Theorem 1.1 does not assume the compactness of the base Y.

For a family of canonically polarized compact Kähler manifolds, we have a fiberwise Kähler-Einstein metric by the similar way. The positivity of the fiberwise Kähler-Einstein metric on a family of compact Kähler manifolds was first studied by Schumacher. In his paper [30], he have proved that the fiberwise Kähler-Einstein metric on a family of canonically polarized compact Kähler manifolds is semi-positive. Moreover he have also proved that it is strictly positive if the family is effectively parametrized. This is equivalent to the semi-positivity or positivity of the relative canonical line bundle of the family, respectively. Păun have shown that if the relative adjoint line bundle is positive on each fiber, then it is semi-positive on the total space by generalizing the method of Schumacher ([28]). Guenancia also have proved the semi-positivity of the fiberwise conic singular Kähler-Einstein metric ([16]). In case of a family of complete Kähler manifolds, Choi have proved that the fiberwise Kähler-Einstein metric on a family of bounded pseudoconvex domains is semi-positive or positive if the total space is pseudoconvex or strongly pseudoconvex, resepctively ([9, 10]).

The proof of Schumacher's theorem starts with the following identity from [31]: For a real $(1,1)$-form τ on X,

$$\tau^{n+1} = c(\tau)\tau^n\sqrt{-1}ds\wedge d\bar{s} \tag{1.1}$$

where τ^n is the n-fold exterior power divided by $n!$. Here $c(\tau)$ is called a *geodesic curvature* of τ. (For the detail, see Section 2.1.) Now suppose that τ is positive-definite on each fiber X_y. Then (1.1) says that τ is semi-positive or positive if and only if $c(\tau) \geq 0$ or $c(\tau) > 0$, respectively. Schumacher have proved that the geodesic curvature of the fiberwise Kähler-Einstein metric on a family of canonically polarized compact Kähler manifolds satisfies a certain second order linear elliptic partial differential equation. This PDE gives a lower bound of the geodesic curvature by the maximum principle or an lower bound estimate on the heat kernel.

However, in case of a Calabi-Yau fibration, the PDE which the geodesic curvature of fiberwise Ricci-flat metric satisfies is of a different type from the previous one. (See Section 4.) In particular, it does not give a lower bound of the geodesic curvature. This is why Schumacher's method does not give positivity or semi-positivity of the fiberwise Ricc-flat metric. But the approximation procedure of complex Monge-Ampere equations, it is possible to obtain a lower bound of the direct image of the fiberwise Ricci-flat metric (see Section 5.) This is the main contribution of this paper.

This difference of the PDEs, which the fiberwise Kähler-Einstein metric on a family of canonically polarized manifolds and the fiberwise Ricc-flat metric on a family of Calabi-Yau manifold satisfy, arises from the difference of complex Monge-Ampère equations which give the Kähler-Einstein metrics. More precisely, the complex Monge-Ampère equation of type:

$$(\omega + dd^c\varphi)^n = e^{\lambda\varphi + f}\omega^n, \tag{1.2}$$

for some constant $\lambda > 0$ and some suitable smooth function f, gives the Käher-Einstein metric on a canonically polarized compact Käher manifold. On the other hand, the complex Monge-Ampère equation of type:

$$(\omega + dd^c\varphi)^n = e^{\tilde{f}}\omega^n \tag{1.3}$$

for some suitable smooth function $\tilde{f}$, gives the Kähler-Einstein (in this case Ricci-flat) metric on a Calabi-Yau manifold. It is remarkable to note that if f and $\tilde{f}$ coincide, then (1.2) converges to (1.3) as $\lambda \to 0$. Then by the a priori estimate for complex Monge-Ampère equation, it is well known that the solutions φ_λ of (1.2) converges to the solution of (1.3) (see Section 3). This is the key obervation of the proof of approximation procedures which we already mentioned.

Although we cannot obtain the positivity of the fiberwise Ricci-flat metric ρ on X, Theorem 1.1 gives a lower bound of ρ which is given by the Green kernel of each fiber and the Weil-Petersson metric on the base Y.

Theorem 1.2. *Let $G_y(z,w)$ be the Green kernel of $-\Delta_{\omega_{KE,y}}$ on X_y which is normalized by*

$$\int_{X_y} G_y(z,w)\, dV_{\omega_{KE,y}}(z) = 0.$$

Let $-K(y)$ with $K(y) \geq 0$ be the lower bound of the Green kernel, i.e.,

$$\inf_{(z,w)\in X_y \times X_y} G_y(z,w) = -K(y).$$

Then $\rho + K(y)\omega^{WP}$ is positive on X, where ω^{WP} is the Weil-Petersson metric on Y. (About the Green kernel, see [2]*.)*

It is remarkable to note that the lower bound of the Green kernel of a compact Kähler manifold is bounded from below by a constant which depends only on the geometry of the compact Kähler manifold, more precisely, the Ricci curvature, the diameter and the volume (Theorem 3.2 in [3]). Since the Ricci curvature of every fiber vanishes and the volume of every fiber is same (see Subsection 3.2), the fiberwise constant $K(y)$ is uniformly bounded from below if the diameter of every fiber is bounded.

In the meantime, the second order elliptic PDE for the geodesic curvature $c(\rho)$ of the fiberwise Ricci-flat metric of a Calabi-Yau fibration gives several informations about Calabi-Yau fibrations. Among them, there is a result about the local triviality of Calabi-Yau fibrations.

Theorem 1.3. *Let $p : X \to Y$ be a smooth family of Calabi-Yau manifolds. Let $E := p_*(K_{X/Y})$ be the direct image bundle of the relative canonical line bundle $K_{X/Y}$. We denote by $\Theta(E)$ the curvature of the natural L^2 metric of E. If $\Theta(E)$ vanishes along a complex curve, then the family is trivial along the complex curve.*

A similar result was obtained by Tosatti in [37] (cf. see also [14]). Jolany informed the author that he also proved Theorem 1.3 and some estiamtes of this paper ([24]).

Acknowlegement. The author happily acknowledges his thanks to Mihai Păun who suggested this problem, shared his ideas and Dano Kim for very helpful discussions. He is also indebted to Hoang Lu Chinh for teaching him the approximation process of complex Monge-Ampère equations, Bo Berndtsson for enlightening discussions about many topics, including the direct image bundles and Griffiths theorem, Henri Guenancia for helpful comments about Proposition 3.7, Long Li for many helpful comments about Green kernels, Philippe Eyssidieux for many helpful comments and discussions and Jean-Pierre Demailly for enlightening comments including the natural hermitian metric on a family of Calabi-Yau manifolds. Finally, he would like to thank Yuxin Ge for informing him the error of the previous version.

2. Preliminaries

Let $p : X^{n+d} \to Y^d$ be a smooth family of Kähler manifolds. Taking a local coordinate $(s^1, \dots, s^d)$ of Y and a local coordinate $(z^1, \dots, z^n)$ of a fiber of p, $(z^1, \dots, z^n, s^1, \dots, s^d)$ forms a local coordinate of X such that under this coordinate, the holomorphic mapping p is locally given by

$$p(z^1, \dots, z^n, s^1, \dots, s^d) = (s^1, \dots, s^d).$$

We call this an *admissible coordinate of* p.

Throughout this paper we use small Greek letters, $\alpha, \beta, \dots = 1, \dots, n$ for indices on $z = (z^1, \dots, z^n)$ and small roman letters, $i, j, \dots = 1, \dots, d$ for indices on $s = (s^1, \dots, s^d)$ unless otherwise specified. For a properly differentiable function f on X, we denote by

$$f_\alpha = \frac{\partial f}{\partial z^\alpha}, \quad f_{\bar\beta} = \frac{\partial f}{\partial z^{\bar\beta}}, \quad \text{and } f_i = \frac{\partial f}{\partial s^i}, \quad f_{\bar j} = \frac{\partial f}{\partial s^{\bar j}}, \tag{2.1}$$

where $z^{\bar\beta}$ and $s^{\bar j}$ mean $\overline{z^\beta}$ and $\overline{s^j}$, respectively. In case $d = 1$, we denote by

$$f_s = \frac{\partial f}{\partial s} \text{ and } f_{\bar s} = \frac{\partial f}{\partial \bar s}.$$

If there is no confusion, we always use the Einstein convention. For simplicity we denote by $v_i := \partial/\partial s^i$. If $d = 1$, then we denote by $v := \partial/\partial s$.

2.1. **Horizontal lifts and geodesic curvatures.** For a complex manifold M, we denote by $T'M$ the complex tangent bundle of type $(1,0)$.

Definition 2.1. Let $V \in T'Y$ and τ be a real $(1,1)$-form on X. Suppose that τ is positive definite on each fiber X_y.

1. A vector field V_τ of type $(1,0)$ is called a *horizontal lift* of V if V_τ satisfies the following:
   (i) $\langle V_\tau, W\rangle_\tau = 0$ for all $W \in T'X_y$,
   (ii) $d\pi(V_\tau) = V$.
2. The *geodesic curvature* $c(\tau)(V)$ of τ along V is defined by the norm of V_τ with respect to the sesquilinear form $\langle\cdot,\cdot\rangle_\tau$ induced by τ, namely,
$$c(\tau)(V) = \langle V_\tau, V_\tau\rangle_\tau .$$

Remark 2.2. Let $(z^1,\dots,z^n,s^1,\dots,s^d)$ be an admissible coordinate of p. Then we can write τ as follows:
$$\tau = \sqrt{-1}\left(\tau_{i\bar{j}}ds^i\wedge ds^{\bar{j}} + \tau_{i\bar{\beta}}ds^i\wedge dz^{\bar{\beta}} + \tau_{\alpha\bar{j}}dz^\alpha\wedge ds^{\bar{j}} + \tau_{\alpha\bar{\beta}}dz^\alpha\wedge dz^{\bar{\beta}}\right).$$
Since τ is positive-definite on each fiber X_y, the matrix $(\tau_{\alpha\bar{\beta}})$ is invertible. We denote by $(\tau^{\bar{\beta}\alpha})$ the inverse matrix. Then it is easy to see that the horizontal lift of $\partial/\partial s^i$ is given as follows.
$$\left(\frac{\partial}{\partial s^i}\right)_\tau = \frac{\partial}{\partial s^i} - \tau_{i\bar{\beta}}\tau^{\bar{\beta}\alpha}\frac{\partial}{\partial z^\alpha},$$
in particular, any horizontal lift with respect to τ is uniquely determined.

On the other hand, the geodesic curvature $c(\tau)(v_i)$ is computed as follows:
$$\begin{aligned} c(\tau)(v_i) &= \langle (v_i)_\tau, (v_i)_\tau\rangle_\tau \\ &= \left\langle \frac{\partial}{\partial s^i} - \tau_{i\bar{\beta}}\tau^{\bar{\beta}\alpha}\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial s^i} - \tau_{i\bar{\delta}}\tau^{\bar{\delta}\gamma}\frac{\partial}{\partial z^\gamma}\right\rangle_\tau \\ &= \tau_{i\bar{i}} - \overline{\tau_{i\bar{\delta}}\tau^{\bar{\delta}\gamma}}\tau_{i\bar{\gamma}} - \tau_{i\bar{\beta}}\tau^{\bar{\beta}\alpha}\tau_{\alpha\bar{i}} + \tau_{i\bar{\beta}}\tau^{\bar{\beta}\alpha}\overline{\tau_{i\bar{\delta}}\tau^{\bar{\delta}\gamma}}\tau_{\alpha\bar{\gamma}} \\ &= \tau_{i\bar{i}} - \tau_{i\bar{\beta}}\tau^{\bar{\beta}\alpha}\tau_{\alpha\bar{i}}, \end{aligned}$$
because τ is a real $(1,1)$-form.

Remark 2.3. The real $(1,1)$-form τ in Definition 2.1 induces a hermitian metric on the relative canonical line bundle $K_{X/Y}$ as follows:

Let $(z^1,\dots,z^n,s^1,\dots,s^d)$ be an admissible coordinate in X. Since τ is positive-definite on each fiber, $(\tau_{\alpha\bar{\beta}})$ is positive-definite. Hence
$$\sum \tau_{\alpha\bar{\beta}}(z,s)dz^\alpha\wedge dz^{\bar{\beta}}$$
gives a Kähler metric on each fiber X_s. It follows that
$$\det(\tau_{\alpha\bar{\beta}}(z,s))^{-1} \tag{2.2}$$
gives a hermitian metric on the relative line bundle $K_{X/Y}$. We denote this metric by $h^\tau_{X/Y}$. The curvature form $\Theta_{h^\tau_{X/Y}}(K_{X/Y})$ of $h^\tau_{X/Y}$ is given by
$$\Theta_{h^\tau_{X/Y}}(K_{X/Y}) = dd^c\log\det(\tau_{\alpha\bar{\beta}}(z,s)).$$

It is obvious that the cuvature is also written as follows:

$$\Theta_{h^{\tau}_{X/Y}}(K_{X/Y}) = dd^c \log\det\left(\tau^n \wedge dV_s\right),$$

where we denote by τ^n the n-fold exterior power divided by $n!$.

Suppose that Y is 1-dimensional. Then it is well known (cf, see [31]) that

$$\tau^{n+1} = c(\tau)\cdot\tau^n \wedge \sqrt{-1}ds\wedge d\bar{s}. \tag{2.3}$$

It follows that if $c(\tau) > 0$ (≥ 0), then τ is a positive (semi-positive) real $(1,1)$-form as τ is positive definite when restricted to X_y. On the other hand, (2.3) says that

$$p_*\tau^{n+1} = \int_{X_s}\tau^{n+1} = \int_{X_s} c(\tau)\cdot\tau^n\wedge\sqrt{-1}ds\wedge d\bar{s}.$$

Hence $p_*\tau^{n+1}$ is positive or semi-positive if and only if $\int_{X_s} c(\tau)\tau^n$ is positive or nonnegative, respectively. For later use, we introduce the following lemma.

Lemma 2.4. *The following identity holds:*

$$i_{v_\tau}\tau = \sqrt{-1}c(\tau)d\bar{s}.$$

Proof. The computation is quite straightforward.

$$\begin{aligned}
i_{v_\tau}\tau &= \sqrt{-1}\left(\tau_{s\bar{s}}d\bar{s} + \tau_{s\bar{\beta}}dz^{\bar{\beta}} - \tau_{s\bar{\beta}}\tau^{\bar{\beta}\alpha}\tau_{\alpha\bar{s}}d\bar{s} - \tau_{s\bar{\delta}}\tau^{\bar{\delta}\alpha}\tau_{\alpha\bar{\beta}}dz^{\bar{\beta}}\right)\\
&= \sqrt{-1}\left(\tau_{s\bar{s}}d\bar{s} - \tau_{s\bar{\beta}}\tau^{\bar{\beta}\alpha}\tau_{\alpha\bar{s}}d\bar{s}\right)\\
&= \sqrt{-1}c(\tau)d\bar{s}.
\end{aligned}$$

This completes the proof. □

2.2. **Kodaira-Spencer classes and Direct image bundles.** Let $p : X \to Y$ be a smooth family of compact Kähler manifolds. We denote the Kodaira-Spencer map for the family $p : X \to Y$ at a given point $y \in Y$ by

$$K_y : T'_yY \to H^1(X_y, T'X_y).$$

The Kodaira-Spencer map is induced by the edge homomorphism for the short exact sequence

$$0 \to T'_{X/Y} \to T'X \to p^*T'Y \to 0.$$

If $V \in T'_yY$ is a tangent vector, and if

$$V + b^\alpha\frac{\partial}{\partial z^\alpha}$$

is any smooth lifting of V along X_y, then

$$\bar\partial\left(V + b^\alpha\frac{\partial}{\partial z^\alpha}\right) = \frac{\partial b^\alpha}{\partial z^{\bar\beta}}\frac{\partial}{\partial z^\alpha}\otimes dz^{\bar\beta}$$

is a $\bar\partial$-closed form on X, which represents $K_y(V)$, i.e.,

$$K_y(V) = \left[\frac{\partial b^\alpha}{\partial z^{\bar\beta}}\frac{\partial}{\partial z^\alpha}\otimes dz^{\bar\beta}\right] \in H^{0,1}(X_y, T'X_y).$$

This cohomology class $K_y(V)$ is called the *Kodaira-Spencer class* of V. The celebrated theorem of Kodaira and Spencer says that if the Kodaira-Spencer class vanishes locally, then the family is locally trivial ([20], see also [19]).

The direct image sheaf $E := p_*(K_{X/Y})$ of $K_{X/Y}$ is defined by the sheaf over Y whose fiber E_y is given by

$$E_y = H^0(X_y, K_{X_y}).$$

It is remarkable to note that this sheaf is indeed a holomorphic vector bundle by the Ohsawa-Takegoshi extension theorem (for more details, see Section 4 in [4]). E is a hermitian vector bundle with L^2 metric defined by following: For $u_y, v_y \in E_y$, define $\langle u_y, v_y \rangle$ by

$$\langle u_y, v_y \rangle_y^2 = \int_{X_y} c_n u_y \wedge \overline{v_y}$$

where $c_n = (\sqrt{-1})^{n^2}$ chosen to make the form positive. The Kodaira-Spencer class acts on $u_y \in E_y$ as follows: Let $k_y(V)$ be any representative of $K_y(V)$, i.e., $T'X_y$-valued $(0,1)$-form in $K_y(V)$, which locally decomposes as

$$k_y = \zeta \otimes w$$

where ζ is a $(0,1)$-form and w is a vector field of type $(1,0)$. Then $k_y(V)$ acts on u_y by

$$k_y(V) \cdot u_y = \zeta \wedge (i_w(u_y)),$$

where i_w is the contraction. This gives a globally defined $\bar{\partial}$-closed form of type $(n-1,1)$ and

$$K_y(V) \cdot u_y := [k_y(V) \cdot u_y] \in H^{n-1,1}(X_y).$$

The following theorem due to Griffiths says the curvature of E is computed in terms of Kodaira-Spencer classes ([18], see also [5]).

Theorem 2.5. *Let $\Theta(E)$ be the curvature of E with L^2-metric. Then for $V \in T'_yY$,*

$$\langle \Theta_{V\bar{V}}(E)u, u \rangle = \|K_y(V) \cdot u\|^2 , \tag{2.4}$$

where $\|K_y(V) \cdot u\|$ is the norm of its unique harmonic representative. It does not depend on the choice of Kähler metric.

3. Approximations of complex Monge-Ampère equations

In this section, we discuss approximations of a solution of complex Monge-Ampère equation (1.3) in terms of the solutions of (1.2). First we consider the approximation on a single compact Kähler manifold. After that, we apply the approximation procedure to a family of complex Monge-Ampère equations. First, we recall the existence and uniqueness theorem of complex Monge-Ampère equations due to Aubin and Yau.

Let (X, ω) be a compact Kähler manifold. Let f be a smooth function on X. The complex Monge-Ampère equation is given by the following:

$$\begin{aligned} (\omega + dd^c\varphi)^n &= e^{\lambda\varphi + f}\omega^n, \\ \omega + dd^c\varphi &> 0. \end{aligned} \tag{3.1}$$

This fully nonlinear complex partial differential equation was first raised by E. Calabi. The easiest case, $\lambda > 0$, was solved by Aubin and Yau, independently ([1, 38]). The next case, $\lambda = 0$, was solved by Yau ([38]). The last case, $\lambda < 0$, is not solved in general. This is why a compact Kähler manifold with positive first Chern class does not have the Kähler-Einstein metric in general (cf, see [36]).

Theorem 3.1. *The following holds:*

1. *(Aubin/Yau) If $\lambda > 0$, then there exists a unique smooth function φ satisfying* (3.1) *for every smooth function $f \in C^\infty(X)$.*
2. *(Yau) If $\lambda = 0$, then there exists a smooth function φ satisfying* (3.1) *for $f \in C^\infty(X)$ such that $\int_X e^f \omega^n = \int_X \omega^n$. Moreover, the solution is unique up to the addition of constants.*

3.1. **Approximation on a compact Kähler manifold.** Let (M, ω) be a compact Kähler manifold and f be a smooth function on M satisfying

$$\int_M e^f \omega^n = \int_M \omega^n.$$

Consider the following complex Monge-Ampère equation:

$$(\omega + dd^c\varphi)^n = e^f \omega^n, \qquad \omega + dd^c\varphi > 0. \tag{3.2}$$

By Theorem 3.1, we already know that there exists a solution which is unique up to addition of constants.

Let $\{f_\varepsilon\}$ be a sequence of smooth functions in M which converges to f as ε goes to 0 in $C^{k,\alpha}(M)$-topology for any $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. We want to approximate a solution of (3.6) by the solutions φ_ε of the following complex Monge-Ampère equations:

$$(\omega + dd^c\varphi_\varepsilon)^n = e^{\varepsilon\varphi_\varepsilon + f_\varepsilon} \omega^n \qquad \omega + dd^c\varphi_\varepsilon > 0, \tag{3.3}$$

as $\varepsilon \to 0$. Note that if $\varepsilon \to 0$, then Equation (3.3) converges to Equation (3.6).

The convention all over this paper is that we will use the same letter "C" to denote a generic constant, which may change from one line to another, but it is independent of the pertinent parameters involved (especially ε).

Proposition 3.2. *For each ε with $0 < \varepsilon \le 1$, let φ_ε be the solution of* (3.3)*. Then for any $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, there exists a constant $C > 0$ which depend only on k, α, the geometry of (M, ω) and the function f such that*

$$\|\varphi_\varepsilon\|_{C^{k,\alpha}(M)} < C.$$

In particular, $\{\varphi_\varepsilon\}$ is a relatively compact subset of $C^{k,\alpha}(M)$ for any positive integer k and $\alpha \in (0, 1)$.

Proof. We may assume that

$$\mathrm{Vol}(M) = \int_X \omega^n = 1.$$

The first step is obtaining a uniform upper bound for φ_ε. For each $\varepsilon > 0$, the solution φ_ε of (3.3) satisfies that

$$1 = \int_M (\omega + dd^c\varphi_\varepsilon)^n = \int_M e^{\varepsilon\varphi_\varepsilon} e^{f_\varepsilon} \omega^n$$

Then Jensen inequality implies that

$$1 \ge \exp\left(\int_M \varepsilon\varphi_\varepsilon e^{f_\varepsilon} \omega^n\right),$$

it is equivalent to

$$\int_M \varphi_\varepsilon e^{f_\varepsilon}\omega^n \le 0.$$

Note that f_ε converges to f as $\varepsilon \to 0$. The Hartogs lemma for quasi-plurisubharmonic functions implies that

$$\sup_M \varphi_\varepsilon < C, \tag{3.4}$$

where C is a constant which depends only on the geometry of (M, ω) and f ([15]). Here we recall the simple version of Kołodziej's uniform estimates (for the general theorem, see [21, 22]).

Theorem 3.3. *Let (M, ω) be a compact Kähler manifold. Assume that φ satisfies the following complex Monge-Ampère equation:*

$$(\omega + dd^c\varphi)^n = F\omega^n,$$
$$\omega + dd^c\varphi > 0.$$

Then

$$\|\varphi\|_{C^0(M)} \le C$$

where $C > 0$ depends only on (M, ω) and on an upper bound for $\|F\|_p$ for some $1 < p \le \infty$.

If we set $F = e^{\varepsilon\varphi_\varepsilon + f_\varepsilon}$, then $|F| < C$ for some $C > 0$ by (3.4). Then it follows from Theorem 3.3 that

$$\|\varphi_\varepsilon\|_{C^0(M)} < C \tag{3.5}$$

for some $C > 0$ which depends only on M and the function f.

The second step is obtaining the Laplacian estimates. We recall the following theorem in [12], which is essentially due to M. Păun ([27], cf. see [32]).

Theorem 3.4. *Let ψ^+ and ψ^- be smooth quasi-plurisubharmonic functions on M. Let $\varphi \in C^\infty(M)$ be such that $\sup_M \varphi = 0$ and*

$$(\omega + dd^c\varphi)^n = e^{\psi^+ - \psi^-}\omega^n.$$

Assume given a constant $C > 0$ such that

$$dd^c\psi^\pm \ge -C\omega, \quad \sup_M \psi^+ \le C.$$

Assume also that the holomorphic bisectional curvature of ω is bounded from below by $-C$. Then there exists $A > 0$ depending on C and $\int_M e^{-2(4C+1)\varphi}\omega^n$ such that

$$0 \le n + \Delta_\omega\varphi \le Ae^{-2\psi^-}.$$

We take $\psi^+ = \varepsilon\varphi_\varepsilon + f_\varepsilon$ and $\psi^- = 0$. Since f_ε converges to f as $\varepsilon \to 0$ and every φ_ε satisfies that

$$dd^c\varphi_\varepsilon > -\omega,$$

it follows from (3.5) that ψ^+ satisfies the hypothesis of Theorem 3.4. Note that $\{\varphi_\varepsilon\}_{0<\varepsilon\le 1}$ is a relatively compact subset of $L^1(X, \omega)$. This implies the Laplacian estimates for φ_ε:

$$|\Delta_\omega\varphi_\varepsilon| < C$$

for some constant $C > 0$ which depends only on the geometry of (M, ω) and the function f by the Uniform Skoda Integrability Theorem due to Zeriahi ([39]).

The final step is $C^{2,\alpha}(M)$-estimate. For $k \geq 2$ and $\alpha \in (0,1)$, the standard Evans-Krylov method ([13, 23]) and Schauder estimates (cf, see [2, 17]) imply

$$\|\varphi_\varepsilon\|_{C^{k,\alpha}(X)} \leq C,$$

where C is a positive constant which depends only on k, α, the geometry of (M, ω) and the function f. This completes the proof. $\square$

Proposition 3.2 implies that there exists a $\hat{\varphi} \in C^\infty(M)$ such that $\varphi_\varepsilon \to \hat{\varphi}$ as $\varepsilon \to 0$ by passing through a subsequence. However, φ_ε converges without choosing a subsequence.

Corollary 3.5. *The solution φ_ε converges to φ which satisfies the following normalization condition*

$$\int_M \varphi e^f \omega^n = 0.$$

Proof. Consider the following complex Monge-Ampère equation:

$$\begin{aligned}(\omega + dd^c\varphi)^n &= e^f \omega^n, \\ \omega + dd^c\varphi &> 0.\end{aligned} \tag{3.6}$$

By Theorem 3.1, we already know that there exists a solution which is unique up to addition of constants. Let φ_0 be the unique solution which satisfies that

$$\int_M \varphi_0 e^f \omega^n = 1.$$

For $0 < \varepsilon \leq 1$, we consider the following equation:

$$\begin{aligned}(\omega + dd^c\varphi_\varepsilon)^n &= e^{\varepsilon\varphi_\varepsilon + f}\omega^n \\ \omega + dd^c\varphi_\varepsilon &> 0,\end{aligned} \tag{3.7}$$

Now we want to show that $\varphi_\varepsilon \to \varphi_0$ in $C^{k,\alpha}(X)$-topology. It is enough to show that

$$\lim_{\varphi \to 0} \int_M \varphi_\varepsilon e^f \omega^n = 0.$$

By Kolodziej's estimate, there exists a uniform constant $C > 0$ such that

$$\|\varphi_\varepsilon\|_{C^{k,\alpha}(M)} < C.$$

It follows that

$$e^{\varepsilon\varphi_\varepsilon} = 1 + \varepsilon\varphi_\varepsilon + o(\varepsilon).$$

On the other hand, we have

$$1 = \int_M \omega^n = \int_M (\omega + dd^c\varphi_\varepsilon)^n = \int_M e^{\varepsilon\varphi_\varepsilon} e^{f_\varepsilon}\omega^n = \int_M (1 + \varepsilon\varphi_\varepsilon + o(\varepsilon))\, e^{f_\varepsilon}\omega^n,$$

so we have

$$\varepsilon \int_M \varphi_\varepsilon e^{f_\varepsilon}\omega^n = o(\varepsilon)\int_X e^{f_\varepsilon}\omega^n.$$

This implies the conclusion. $\square$

3.2. **Approximation on a family of complex Monge-Ampére equations.** Let $p : X^{n+d} \to Y^d$ be a smooth family of compact Kähler manifolds and ω be a fixed Kähler form on X. Let ξ be a differential form of degree $2n + r$ on X. Then the fiber integral is a differential form of degree r on Y, which is defined as follows: Fix a point $y \in Y$ and let $(U, s = (s^1, \ldots, s^d))$ be a coordinate centered at y such that there exists a C^∞ trivialization of the family:

$$\Phi : X_0 \times U \to p^{-1}(U).$$

In an admissible coordinate (z, s), the pull-back $\Phi^*\xi$ is of the form

$$\sum \xi_k(z, s) dV_z \wedge d\sigma^{k_1} \wedge \cdots \wedge d\sigma^{k_r},$$

where the σ^{k_j} run through the real and imaginay parts of s^j and dV_z denotes the relative Euclidean volume form. Now the fiber integral is defined by

$$\int_{X/Y} \xi = \int_{X_0 \times Y/Y} \Phi^*\xi = \sum \left(\int_{X_s} \xi_k(z, s) dV_z \right) d\sigma^{k_1} \wedge \cdots \wedge d\sigma^{k_r}.$$

Note that this definition is independent of the choice of coordinates and differentiable trivializations. The fiber integral coincides with the push-forward of the corresponding current. Hence, if ξ is a differentiable form of type $(n+r, n+s)$, then the fiber integral is of type (r, s). In particular, if ξ be a differentiable form of type (n, n) on X, then $\int_{X_s} \xi$ is a smooth function on Y. Moreover, we have the following properties (for the details, see [30].):

(i) Fiber integration coincides with the push forward of a form, which is defined as follows: For a form ξ on X, $p_*\xi$ is defined by the form on Y which satisfies

$$\int_Y (p_*\xi) \wedge \zeta = \int_X \xi \wedge (p^*\zeta)$$

for any form ζ on Y.

(ii) Fiber integration commutes with taking exterior derivatives:

$$d \int_{X_s} \xi = \int_{X_s} d\xi$$

(iii) For a smooth form ξ of type (n, n),

$$\frac{\partial}{\partial s^i} \int_{X_s} \xi = \int_{X_s} L_V(\xi)$$

for any smooth lifting V of $\partial/\partial s^i$ on X.

Note that the volume of a fiber does not change, namely, (ii) implies that

$$d\mathrm{Vol}_{\omega|_{X_s}}(X_s) = d \int_{X_s} \omega^n = \int_{X_s} d\omega^n = 0.$$

Hence we may assume that $\mathrm{Vol}_{\omega|_{X_y}}(X_y) = 1$ for every $y \in Y$. The third property (iii) will be used in Section 6.

From now on, we consider a smooth family $p : X \to \mathbf{D}$ of compact Kähler manifolds over the unit disc $\mathbf{D}$ in $\mathbb{C}$. Let ω be a Kähler form on X. Under an admissible coordinate $(z^1, \ldots, z^n, s)$ in X, ω is written as follows:

$$\omega = \sqrt{-1} \left(g_{s\bar{s}} ds \wedge d\bar{s} + g_{s\bar{\beta}} ds \wedge dz^{\bar{\beta}} + g_{\alpha\bar{s}} dz^\alpha \wedge d\bar{s} + g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} \right). \tag{3.8}$$

For $0 < \varepsilon \leq 1$, let $\{f_\varepsilon\}$ be a sequence of smooth functions on X. We consider the following fiberwise complex Monge-Ampère equations:

$$
\begin{aligned}
(\omega_y + dd^c\varphi_y)^n &= e^{\varepsilon\varphi_y + f_\varepsilon|_{X_y}}(\omega_y)^n, \\
\omega_y + dd^c\varphi_y &> 0
\end{aligned}
\tag{3.9}
$$

on X_y for $y \in \mathbf{D}$. Theorem 3.1 implies that for each y, there exists a unique solution of (3.9), call it $\varphi_{y,\varepsilon} \in C^\infty(X_y)$. It is remarkable to note that the function φ_ε defined by

$$\varphi_\varepsilon(x) = \varphi_{y,\varepsilon}(x),$$

where $y = p(x)$, is a smooth function on X. This follows from the openness analysis of the continuity method for complex Monge-Ampère equations and the implicit function theorem ([38]). By Section 3.1, there exists a constant $C_y > 0$ such that

$$\|\varphi_\varepsilon\|_{C^{k,\alpha}(X_y)} \leq C_y \tag{3.10}$$

where C_y does not depend on ε. Since we are now considering a local property on y, we may assume that $C = C_y$ does not depend on y.

In this section, we consider the $C^{k,\alpha}$-estimates for $V\varphi_\varepsilon$ and $\bar{V}V\varphi_\varepsilon$ on a fixed fiber X_y, where V is any smooth lifting of $\partial/\partial s$ written as follows:

$$V = \frac{\partial}{\partial s} + a_s{}^\gamma \frac{\partial}{\partial z^\gamma}.$$

Before going further, we introduce the following proposition.

Proposition 3.6. *Let (X, ω) be a compact Kähler manifold. Let $\{\rho_\varepsilon\}_{\varepsilon \in I}$ be a family of Kähler metrics on X which are uniformly equivalent to ω, i.e., there exists a constant $C_1 > 0$ such that*

$$\frac{1}{C_1}\omega < \rho_\varepsilon < C_1\omega \quad \text{for all} \quad \varepsilon \in I.$$

Let u_ε be a solution of the following PDE:

$$-\Delta_{\rho_\varepsilon} u_\varepsilon + \varepsilon u_\varepsilon = R_\varepsilon, \tag{3.11}$$

where R_ε is a smooth function on X with

$$\|R_\varepsilon\|_{C^{k,\alpha}(X)} < C_2.$$

Suppose that

$$\left| \int_X u_\varepsilon \omega^n \right| < C_3.$$

Then there exists a uniform constant $C > 0$ which depends only on C_1, C_2, C_3 and the geometry of (X, ω) such that

$$\|u_\varepsilon\|_{C^{k,\alpha}(X)} < C.$$

Proof. In this proof, we shall use the Schauder estimate, Poincaré inequality and Sobolev inequality with respect to the Kähler metric ρ_ε (cf, see [17, 2]). It is remakable to note that the constants in those inequalities do not depend on $\varepsilon \in I$ since all ρ_ε are uniformly equivalent to ω. If we have the uniform estimate, i.e., C^0-estiamte of u, then Schauder estimate completes the proof.

The Poincaré inequality says that there exists a constant C which depends only on C_1 and the geometry of (M,ω) such that

$$\left\| u_\varepsilon - \int_X u_\varepsilon {\rho_\varepsilon}^n \right\|_{L^2_{\rho_\varepsilon}(X)} < C \left\| D u_\varepsilon \right\|_{L^2_{\rho_\varepsilon}(X)},$$

where D is a total derivative. It follows from the assumption that

$$\| u_\varepsilon \|_{L^2_{\rho_\varepsilon}(X)} < C \left\| D u_\varepsilon \right\|_{L^2_{\rho_\varepsilon}(X)} + C_1 C_3.$$

On the other hand, multiplying u_ε to (3.11) and integrating it with respect to $(\rho_\varepsilon)^n$, we have

$$\| D u_\varepsilon \|^2_{L^2_{\rho_\varepsilon}(X)} + \varepsilon \| u_\varepsilon \|^2_{L^2_{\rho_\varepsilon}(X)} = \int_X R_\varepsilon u_\varepsilon {\rho_\varepsilon}^n.$$

The Hölder inequality says that

$$\| D u_\varepsilon \|^2_{L^2_{\rho_\varepsilon}(X)} \leq \| R_\varepsilon \|_{L^2_{\rho_\varepsilon}(X)} \| u_\varepsilon \|_{L^2_{\rho_\varepsilon}(X)} \tag{3.12}$$

Combining the two equations, there exists a uniform constant C which depends only on C_1, C_2, C_3 and the geometry of (X,ω) such that

$$\| u_\varepsilon \|_{L^2_{\rho_\varepsilon}(X)} < C.$$

Now we follow the Moser iteration. Multiplying (3.16) by $|u_\varepsilon|^{2p-1} \cdot u_\varepsilon / |u_\varepsilon|$ and integrating it, we have

$$\frac{2p-1}{p^2} \int_{X_y} |D |u_\varepsilon|^p|^2 \omega^n + \varepsilon \int_X |u_\varepsilon|^{2p} \omega^n = \int_X R_\varepsilon u_\varepsilon \omega^n.$$

The Sobolev inequality says that

$$\| |u_\varepsilon|^p \|^2_{L^{2n/(n-1)}_{\rho_\varepsilon}(X)} \leq C \left(\| |u_\varepsilon|^p \|_{L^2_{\rho_\varepsilon}(X)} + \| D |u_\varepsilon|^p \|_{L^2_{\rho_\varepsilon}(X)} \right)$$

for $p \geq 1$ ([2]). Combining two equations, we have

$$\| u_\varepsilon \|_{L^{2p \cdot \frac{n}{n-1}}_{\rho_\varepsilon}(X)} \leq (Cp)^{1/p} \| u_\varepsilon \|_{L^{2p}_{\rho_\varepsilon}(X)}$$

for $p \geq 1$. The uniform estimate is obtained by the Moser iteration method (cf, see [17]). Indeed, set

$$p_1 = 1, \quad p_k = \left(\frac{n}{n-1} \right)^k.$$

Then it follows that

$$\| u \|_{L^\infty(X)} = \lim_{k \to \infty} \| u_\varepsilon \|_{L^{2p_k}_{\rho_\varepsilon}(X)} \leq \prod_{k=1}^n (C p_k)^{1/p_k} \| u_\varepsilon \|_{L^2_{\rho_\varepsilon}(X)}.$$

This completes the proof. □

Proposition 3.7. *Suppose that there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\left| \int_{X_y} (V \varphi_\varepsilon)(\omega_y)^n \right| < C_1$$

and

$$\| V f_\varepsilon \|_{C^{k,\alpha}(X_y)} < C_2.$$

Then there exits a constant C which depends only on the constants C_1, C_2, the lift V and the geometry of (X_y, ω_y) such that

$$\|V\varphi_\varepsilon\|_{C^{k,\alpha}(X_y)} < C$$

for $0 < \varepsilon \le 1$. In particular, $\{V\varphi_\varepsilon\}_{0<\varepsilon\le 1}$ is a relatively compact subset in $C^{k,\alpha}(X_y)$ for any $k \in \mathbb{N}$ and $\alpha \in (0,1)$.

Proof. We denote by $\rho_\varepsilon = \omega + dd^c\varphi_\varepsilon$. Note that Proposition 3.2 implies that there exists a uniform constant $C > 0$ such that

$$\frac{1}{C}\omega_y < \rho_\varepsilon|_{X_y} < C\omega_y, \tag{3.13}$$

for $0 < \varepsilon \le 1$. Under an admissible coordinate $(z^1, \dots, z^n, s)$, the first equation of (3.9) is written as follows:

$$\det(g_{\alpha\bar\beta} + (\varphi_\varepsilon)_{\alpha\bar\beta}) = e^{\varepsilon\varphi_\varepsilon + f_\varepsilon}\det(g_{\alpha\bar\beta}) \tag{3.14}$$

on each X_y. Taking logarithm of (3.14) and differentiating it with respect to V, we have

$$(\rho_\varepsilon)^{\alpha\bar\beta} V\left(g_{\alpha\bar\beta} + (\varphi_\varepsilon)_{\alpha\bar\beta}\right) = \varepsilon V\varphi_\varepsilon + Vf_\varepsilon + g^{\alpha\bar\beta}V\left(g_{\alpha\bar\beta}\right).$$

For a smooth function ξ, we denote by

$$\begin{aligned}[V,\xi]_{\alpha\bar\beta} &= V(\xi_{\alpha\bar\beta}) - (V\xi)_{\alpha\bar\beta} \\ &= -a_s{}^\gamma{}_{\alpha\bar\beta}\xi_\gamma - a_s{}^\gamma{}_\alpha\xi_{\gamma\bar\beta} - a_s{}^\gamma{}_{\bar\beta}\xi_{\alpha\gamma}.\end{aligned}$$

It is remarkable to note that $[V,\xi]_{\alpha\bar\beta}$ does not include s-derivative of ξ. Then it follows that

$$\begin{aligned}-\Delta_{\rho_\varepsilon|_{X_y}}(V\varphi_\varepsilon) + \varepsilon(V\varphi_\varepsilon) = &- Vf_\varepsilon - g^{\alpha\bar\beta}V\left(g_{\alpha\bar\beta}\right) \\ &+ (\rho_\varepsilon)^{\alpha\bar\beta}\left(V\left(g_{\alpha\bar\beta}\right) + [V,\varphi_\varepsilon]_{\alpha\bar\beta}\right)\end{aligned} \tag{3.15}$$

on each fiber X_y, where $\Delta_{\rho_\varepsilon|_{X_y}}$ is the Laplace-Beltrami operator on X_y with respect to $\rho_\varepsilon|_{X_y}$. Here $(V\varphi_\varepsilon)$ and (Vf_ε) mean that

$$V\varphi_\varepsilon = (V\varphi_\varepsilon)|_{X_y} \quad \text{and} \quad Vf_\varepsilon = (Vf_\varepsilon)|_{X_y}.$$

From now on, when we think about a family of PDEs, we omit the subsrcript X_y in the Laplace-Beltrami opertor, i.e., we write as follows:

$$\Delta_{\rho_\varepsilon} = \Delta_{\rho_\varepsilon|_{X_y}}.$$

Equation (3.15) says that the right hand side of (3.15) is a globally defined function on X_y, call it R_ε. Then we have

$$-\Delta_{\rho_\varepsilon}(V\varphi_\varepsilon) + \varepsilon(V\varphi_\varepsilon) = R_\varepsilon. \tag{3.16}$$

This is a second order elliptic partial differential equation with the hypotheses in Proposition 3.6. This completes the proof. □

Proposition 3.8. *Under the assumption in Proposition 3.7, suppose that there exists a constant $C_3 > 0$ and C_4 such that*

$$\left|\int_{X_y}\left(\bar V V\varphi_\varepsilon\right)(\omega_y)^n\right| < C_3$$

and

$$\left\|\bar V V f_\varepsilon\right\|_{C^{k,\alpha}(X_y)} < C_4.$$

Then there exits a constant C which depends only on constants C_1, C_2, C_3, C_4, the lift V and the geometry of (X_y, ω_y) such that

$$\left\|\bar{V} V \varphi_\varepsilon\right\|_{C^{k,\alpha}(X_y)} < C$$

for $0 < \varepsilon \leq 1$. In particular, $\{\bar{V} V \varphi_\varepsilon\}_{0<\varepsilon\leq 1}$ is a relatively compact subset in $C^{k,\alpha}(X_y)$ for any $k \in \mathbb{N}$ and $\alpha \in (0,1)$.

Proof. Differentiating (3.16) with respect to $\bar{V}$, we have

$$-\Delta_{\rho_\varepsilon}\left(\bar{V} V \varphi_\varepsilon\right) + \varepsilon\left(\bar{V} V \varphi_\varepsilon\right) = \bar{V}\left((\rho_\varepsilon)^{\bar{\beta}\alpha}\right) \cdot (V\varphi_\varepsilon)_{\alpha\bar{\beta}} + (\rho_\varepsilon)^{\bar{\beta}\alpha}[\bar{V}, V\varphi_\varepsilon]_{\alpha\bar{\beta}} + \bar{V}(R_\varepsilon).$$

Since $\|\varphi_\varepsilon\|_{C^{k,\alpha}(X_y)}$ and $\|V\varphi_\varepsilon\|_{C^{k,\alpha}(X_y)}$ are bounded, the same argument in the proof of Propostion 3.7 says this PDE satisfies the hypotheses in Proposition 3.6. This completes the proof. □

4. Fiberwise Ricci-flat metrics on Calabi-Yau fibrations

In this section, we discuss the properties of the fiberwise Ricci-flat metric ρ. We first discuss a partial differential equation which the geodesic curvature $c(\rho)$ satisfies and several applications of this PDE.

Let $p : X \to Y$ be a smooth family of Calabi-Yau manfiolds and ω be a Kähler form on X. We write ω like as (3.8). Since every fiber X_y is a Calabi-Yau manifold, the first Chern class $c_1(X_y)$ vanishes for each fiber X_y. Since $c_1(X_y)$ is represented by the Ricci form of ω_y, we know that

$$\left[-dd^c \log\det(g_{\alpha\bar{\beta}}(\cdot, y))\right] = 0.$$

By the dd^c-lemma, there exists a unique function $\eta_y \in C^\infty(X_y)$ such that

- $dd^c\eta_y = dd^c \log\det(g_{\alpha\bar{\beta}})$ and
- $\int_{X_y} e^{\eta_y}(\omega_y)^n = \int_{X_y}(\omega_y)^n$.

For each $y \in Y$, there exists a unique solution $\varphi_y \in C^\infty(X_y)$ of the following complex Monge-Ampère equation on each fiber X_y:

$$\begin{aligned} (\omega_y + dd^c\varphi_y)^n &= e^{\eta_y}(\omega_y)^n, \\ \omega_y + dd^c\varphi_y &> 0, \end{aligned} \tag{4.1}$$

which is normalized by

$$\int_{X_y} \varphi_y e^{\eta_y}(\omega_y)^n = 0.$$

Then it is easy to see that $\omega_y + dd^c\varphi_y$ is the Ricci-flat Kähler metric on X_y. As we already mentioned, we can consider φ as a smooth function on X by letting $\varphi(x) = \varphi_y(x)$ where $y = p(x)$. Define a real $(1,1)$-form ρ on X by

$$\rho = \omega + dd^c\varphi.$$

Since $e^{\eta_y}(\omega_y)^n = (\omega_{KE,y})^n$, this is the fiberwise Ricci-flat metric in Theorem 1.1.

Since every fiber X_y is Calabi-Yau, K_{X_y} is a trivial line bundle for every $y \in Y$. Hence the direct image bundle $E = p_*(K_{X/Y})$ is a line bundle over Y. Take an admissible coordinate system $(z^1, \ldots, z^n, s^1, \ldots, s^d)$ in X. Let u be a local holomorphic section of E over an open set $U \subset Y$. (Shrinking U if necessary, $s = (s^1, \ldots, s^d)$ can

be considered as a local coordinate in U.) Since E is a line bundle, the curvature of $(E, \|\cdot\|)$ is given by

$$\Theta(E) = -dd^c \log \|u\|_s .$$

We say that $\mathbf{u}$ is a representative of u if $\mathbf{u}$ is an $(n,0)$-form on $p^{-1}(U)$, such that $\mathbf{u}$ restricts to u_s on fibers X_s, i.e.,

$$\iota_s^*(\mathbf{u}) = u_s$$

where ι_s is the natural inclusion map from X_s to X (for more details, see [4, 5]). The representative is not uniquely determined, but any two representatives are differ from $ds \wedge v$ for some $(n-1,0)$-form v. Hence if we denote by $u \wedge \overline{u} \wedge dV_s := \mathbf{u} \wedge \overline{\mathbf{u}} \wedge dV_s$, where $dV_s = c_d ds \wedge d\bar{s}$, then it does not depend on the choice of the representative. Moreover, it also follows that

$$\|u\|_s^2 = c_n \int_{X_s} u \wedge \overline{u} = c_n \int_{X_s} \mathbf{u} \wedge \overline{\mathbf{u}}$$

for any representative $\mathbf{u}$ of u. In terms of u, the function η is written explicitly:

Proposition 4.1. *On $p^{-1}(U)$, η is written as follows:*

$$\eta(z,s) = -\log \frac{\omega^n \wedge dV_s}{c_n u \wedge \overline{u} \wedge dV_s} - \log \|u\|_s^2 . \tag{4.2}$$

In particular, we have the following:

$$dd^c \eta = -\Theta_{h^\omega_{X/Y}}(K_{X/Y}) + \Theta(E).$$

Proof. Let $\mathbf{u}$ be a representative of u. Denote the right hand side of (4.2) by $\tilde{\eta}$. It is enough to show the following:

1. $\int_{X_s} e^{\tilde{\eta}} (\omega_s)^n = 1$.
2. $dd^c \tilde{\eta}|_{X_s} = -dd^c \log \det(g_{\alpha\bar{\beta}})|_{X_s}$.

First we compute

$$\int_{X_s} e^{\tilde{\eta}} (\omega_s)^n = \int_{X_s} \left[\exp \left(-\log \frac{\omega^n \wedge dV_s}{c_n \mathbf{u} \wedge \overline{\mathbf{u}} \wedge dV_s} - \log \|u\|_s^2 \right) \right] (\omega_s)^n$$

If we write $dz = dz^1 \wedge \cdots \wedge dz^n$, then

$$(\omega_s)^n = \det(g_{\alpha\bar{\beta}}) c_n dz \wedge d\bar{z} \quad \text{and} \quad \mathbf{u}|_{X_s} = \hat{u}(z,s) dz$$

for some local holomorphic function $\hat{u}(z,s)$. It follows that

$$\begin{aligned}
\int_{X_s} e^{\tilde{\eta}} (\omega_s)^n &= \int_{X_s} \exp \left(-\log \frac{\det(g_{\alpha\bar{\beta}})}{c_n |\hat{u}(z,s)|^2} - \log \|u\|_s^2 \right) (\omega_s)^n \\
&= \frac{1}{\|u\|_s^2} \int_{X_s} \frac{c_n |\hat{u}(z,s)|^2}{\det(g_{\alpha\bar{\beta}})} \det(g_{\alpha\bar{\beta}}) dz \wedge d\bar{z} \\
&= \frac{1}{\|u\|_s^2} \cdot c_n \int_{X_s} \frac{|\hat{u}(z,s)|^2}{\det(g_{\alpha\bar{\beta}})} \det(g_{\alpha\bar{\beta}}) dz \wedge d\bar{z} \\
&= \frac{1}{\|u\|_s^2} \cdot c_n \int_{X_s} \hat{u}(z,s) dz \wedge \overline{\hat{u}(z,s) dz} \\
&= \frac{1}{\|u\|_s^2} \cdot c_n \int_{X_s} \mathbf{u} \wedge \overline{\mathbf{u}} = 1
\end{aligned}$$

This yields the first assertion. For the second assertion,

$$
\begin{aligned}
dd^c\tilde{\eta}|_{X_s} &= -dd^c\left(\log\frac{\omega^n\wedge dV_s}{c_n u\wedge\overline{u}\wedge dV_s}+\log\|u_s\|^2\right)\Big|_{X_s}\\
&= -dd^c\left(\log\det(g_{\alpha\bar\beta})+\log|\hat{u}(z,s)|^2\right)\Big|_{X_s}\\
&= -dd^c\log\det(g_{\alpha\bar\beta})|_{X_s}.
\end{aligned}
$$

For the second assertion,

$$
\begin{aligned}
dd^c\eta &= -dd^c\log\frac{\omega^n\wedge dV_s}{c_n u\wedge\overline{u}\wedge dV_s}-dd^c\log\|u\|_s^2\\
&= -dd^c\log\frac{\det(g_{\alpha\bar\beta})c_n dz\wedge d\bar z\wedge dV_s}{|\hat{u}(z,s)|^2 c_n dz\wedge d\bar z\wedge dV_s}-dd^c\log\|u\|_s^2\\
&= -dd^c\log\det\left(g_{\alpha\bar\beta}(z,s)\right)+dd^c\log|\hat{u}(z,s)|-dd^c\log\|u\|_s^2\\
&= -\Theta_{h^\omega_{X/Y}}(K_{X/Y})+dd^c\log|\hat{u}(z,s)|^2+\Theta(E)\\
&= -\Theta_{h^\omega_{X/Y}}(K_{X/Y})+\Theta(E).
\end{aligned}
$$

This completes the proof. □

Since ρ is positive-definite on each fiber, it induces a hermitian metric $h^\rho_{X/Y}$ on $K_{X/Y}$ as in Remark 2.3. The curvature of $h^\rho_{X/Y}$ is computed by Proposition 4.1 as follows:

$$
\begin{aligned}
\Theta_{h^\rho_{X/Y}}(K_{X/Y}) &= dd^c\log\left(\rho^n\wedge\sqrt{-1}ds\wedge d\bar s\right)\\
&= dd^c\log\left((\omega+dd^c\varphi)^n\wedge\sqrt{-1}ds\wedge d\bar s\right)\\
&= dd^c\log\left(e^\eta\omega^n\wedge\sqrt{-1}ds\wedge d\bar s\right)\\
&= dd^c\eta+\Theta_{h^\omega_{X/Y}}(K_{X/Y})-dd^c\log\omega^n\wedge dV_s\\
&= -\Theta_{h^\omega_{X/Y}}(K_{X/Y})+\Theta(E)+\Theta_{h^\omega_{X/Y}}(K_{X/Y})\\
&= \Theta(E).
\end{aligned}
$$

Here $\Theta(E)$ means $p^*\Theta(E)$. This formula enables us to compute the Laplacian of $c(\rho)$ on each fiber X_y:

Theorem 4.2. *Let $V\in T_yY$. Then the following PDE holds on X_y:*

$$
-\Delta_\rho c(\rho)(V)=\left|\bar\partial V_\rho\right|_\rho^2-\Theta_{V\bar V}(E). \tag{4.3}
$$

The computation is quite straight forward. Later, we will prove this for more general situation (See Theorem 5.1).

Remark 4.3. To show that $p_*\rho^{n+1}$ is positive on Y, it is enough to consider a Calabi-Yau fibration over the unit disc by the following:

1. Let σ_1 and σ_2 be real $(1,1)$-forms on Y. Suppose that
$$p_*(\sigma_1|_{\gamma(\mathbf{D})})^{n+1}\geq p_*(\sigma_2|_{\gamma(\mathbf{D})})^{n+1}$$
for each holomorphic disc $\gamma^{n+1}:\mathbf{D}\to Y$. Then we have $p_*(\sigma_1)^{n+1}\geq p_*(\sigma_2)^{n+1}$ on X.
2. Every computation concerning the positivity of $p_*\rho^{n+1}$ is local in s-variable, which is a local coordinate in Y.

Therefore we only consider a famliy of Calabi-Yau manifolds over the unit disc in $\mathbb{C}$ as long as we are interested in positivity properties of $p_*\rho^{n+1}$. In this case, (4.3) turns out to be

$$-\Delta_\rho c(\rho) = \left|\bar\partial v_\rho\right|^2_\rho - \Theta_{s\bar s}(E), \tag{4.4}$$

where $v = \partial/\partial s$ and $\Theta_{s\bar s}(E) = \Theta(E)(v, \bar v)$. As we mentioned in Section 2.1, the positivity of $p_*\rho^{n+1}$ is equivalent to $\int_{X_y} c(\rho)\rho^n > 0$.

Remark 4.4. In case of a family of canonically polarized compact complex manifolds $p : X \to \mathbf{D}$, Schumacher have proved that the geodesic curvature $c(\tilde\rho)$ of the form $\tilde\rho$, which is induced by the fiberwise Kähler-Einstein metrics of Ricci curvature -1, satisfies the following PDE:

$$-\Delta_\rho c(\tilde\rho) + c(\tilde\rho) = \left|\bar\partial v_{\tilde\rho}\right|^2_{\tilde\rho} \tag{4.5}$$

for each fiber X_y ([30]). This PDE gives a lower bound of $c(\tilde\rho)$ directly by the maximum principle. (More precise lower bound is also obtained using heat kernel estimates by Schumacher.) Hence the fiberwise Kähler-Einstein form $\tilde\rho$ is a semi-positive metric on X. However (4.4) does not gives a lower bound by the maximum principle.

It is worthwhile to note that the Weil-Petersson metric on the moduli space of canonically polarized manifolds is expressed by the fiberwise Kähler-Einstein metric $\tilde\rho$. More precisely, it follows from (4.5) and (2.3) the Weil-Petersson metric ω_{WP} is written by

$$\omega_{WP} = \int_{X_s} \tilde\rho^{n+1}. \tag{4.6}$$

In case of the moduli space of polarized Calabi-Yau manifolds, our fiberwise Ricci-flat metric does not give such kind of identity. Recently, Braun proved that there exists a Kähler form ω_{SRF} on a family of polarized Calabi-Yau manifolds with vanishing first betti number such that the restriction of the Kähler form on each fiber is Ricc-flat metric and it satisfies (4.6) ([7]).

In the last of this section, we discuss some applications of Theorem 4.2. The *Weil-Petersson form* ω^{WP} of a family $p : X \to Y$ is a real $(1,1)$-form on Y which is induced by the following norm:

$$\|V\|^2_{WP} = \int_{X_y} \left\|\bar\partial V_\rho\right\|^2_\rho dV_\rho.$$

Proposition 4.5. *For $V \in T'Y$, the following holds:*

$$\left\|\bar\partial V_\rho\right\|^2_\rho = \Theta_{V\bar V}(E).$$

In particular, $\omega^{WP} = \Theta(E)$.

Proof. Integrating (4.3) on X_y gives the conclusion. □

Proposition 4.6. *$\bar\partial V_\rho \cdot u_y$ is the harmonic representative of the cohomology class $K_y(V) \cdot u_y$ with respect to $\rho|_{X_y}$.*

Proof. Since E is a line bundle, Griffiths' theorem implies that

$$\Theta_{V\bar V}(E) = \frac{\|K_y(V) \cdot u_y\|^2}{\|u_y\|^2}.$$

Note that

$$\bar\partial V_\rho \in K_y(V).$$

It follows that

$$\frac{\|K_y(V)\cdot u_y\|^2}{\|u_y\|^2} \le \frac{\left\|\bar\partial V_\rho \cdot u_y\right\|^2}{\|u_y\|^2}.$$

The following lemma is well known (cf, see [29]).

Lemma 4.7. *Let (X,ω) be a Calabi-Yau manifold. Let u be a non-vanishing holomorphic n-form on X such that*

$$\|u\|_\omega^2 := \int_X |u|_\omega^2 \; dV_\omega = \int_X dV_\omega = 1.$$

Denote by $A^{(p,q)}(E)$ the space of smooth (p,q)-forms with values in E. Define a map

$$T_u : A^{(0,1)}(T'X) \to A^{(n-1,1)}(X)$$

by $T_u(V) = V\cdot u$. Then T_u is an isometry with respect to the pointwise scalar product induced by ω.

Hence Propositioin 4.4 implies that

$$\left\|\bar\partial V_\rho\right\|_\rho^2 = \Theta_{V\bar V}(E) = \frac{\|K_y(V)\cdot u_y\|^2}{\|u_y\|^2} \le \frac{\left\|\bar\partial V_\rho \cdot u_y\right\|^2}{\|u_y\|^2} = \left\|\bar\partial V_\rho\right\|_\rho^2.$$

It follows that $\bar\partial V_\rho \cdot u_y$ is the harmonic representative with respect to $\rho|_{X_y}$ of $K_y(V)\cdot u_y$. This completes the proof. ☐

Proposition 4.8. *Let $p : X \to Y$ be a Calabi-Yau fibration. If the curvature of the direct image bundle $p_*(K_{X/Y})$ vanishes along a complex curve, then the fibration is trivial along the complex curve.*

Proof. Denote by γ the complex curve in Y. Then $p|_\gamma : X_\gamma \to \gamma$ is a Calabi-Yau fibration over a 1-dimensional base. If we take s be a holomorphic coordinate of γ, then we have Equation (4.4) on each fiber X_y for $y \in \gamma$. By the Hypothesis, $\Theta_{s\bar s}(E)$ vanishes on γ. Proposition 4.5 implies that v_ρ is a holomorphic vector field on X_γ. The flow of v_ρ makes X_γ a trivial fibration. ☐

5. Proof of Theorem 1.1 and Theorem 1.2

In this section we shall prove the main theorem. As we mentioned in Remark 4.3, it is enough to show that $\int_{X/\mathbf{D}} c(\rho)\rho^n \ge 0$ for a family of Calabi-Yau manifolds over the unit disc in $\mathbb{C}$.

Let $p : X \to \mathbf{D}$ be a smooth family of Calabi-Yau manifolds. For each ε with $0 < \varepsilon \le 1$, we consider the following fiberwise complex Monge-Ampère equation on each fiber X_y:

$$\begin{aligned} (\omega_y + dd^c\varphi_y)^n &= e^{\varepsilon\varphi_y} e^{\eta_y} (\omega_y)^n \;\text{ and} \\ \omega_y + dd^c\varphi_y &> 0, \end{aligned} \tag{5.1}$$

where η is defined in Section 4. Theorem 3.1 implies that there exists a unique solution $\varphi_{y,\varepsilon} \in C^\infty(X_y)$ of (5.1). As we mentioned, we can consider φ_ε as a smooth function on X by letting $\varphi_\varepsilon(x) := \varphi_{y,\varepsilon}(x)$, where $y = p(x)$. We consider next the $(1,1)$-form

$$\rho_\varepsilon := \omega + dd^c\varphi_\varepsilon \tag{5.2}$$

on the manifold X. Since ρ_ε is positive definite when restricted to X_y, it induces a hermitian metric $h^{\rho_\varepsilon}_{X/Y}$ on the bundle $K_{X/Y}|_{X_0}$. By Proposition 4.1, the curvature is computed as follows:

$$\begin{aligned}\Theta_{h^{\rho_\varepsilon}_{X/Y}}(K_{X/Y}) &= dd^c \log\left((\rho_\varepsilon)^n \wedge \sqrt{-1}ds\wedge d\bar{s}\right)\\ &= dd^c \log\left((\omega + dd^c\varphi_\varepsilon)^n \wedge \sqrt{-1}ds\wedge d\bar{s}\right)\\ &= dd^c \log\left(e^{\varepsilon\varphi_\epsilon+\eta}\omega^n \wedge \sqrt{-1}ds\wedge d\bar{s}\right)\\ &= dd^c\eta + \varepsilon dd^c\varphi_\varepsilon + \Theta_{h^{\omega}_{X/Y}}(K_{X/Y})\\ &= \Theta(E) + \varepsilon dd^c\varphi_\varepsilon.\end{aligned}$$

From (5.2), we have $dd^c\varphi_\varepsilon = \rho_\varepsilon - \omega$, it follows that

$$\Theta_{h^{\rho_\varepsilon}_{X/Y}}(K_{X/Y}) = \varepsilon\rho_\varepsilon - \varepsilon\omega + \Theta(E) \tag{5.3}$$

in the other expression,

$$\varepsilon\rho_\varepsilon = \varepsilon\omega + \Theta_{h^{\rho_\varepsilon}_{X/Y}}(K_{X/Y}) - \Theta(E).$$

Our next claim is the geodesic curvature $c(\rho_\varepsilon)$ satisfies a certain elliptic partial differential equation of second order on each fiber X_y.

Under an admissible coordinate $(z^1,\dots,z^n,s)\in X$, ρ_ε is written as follows:

$$\rho_\varepsilon = \sqrt{-1}\left((h_\varepsilon)_{s\bar{s}}ds\wedge d\bar{s} + (h_\varepsilon)_{s\bar{\beta}}ds\wedge dz^{\bar{\beta}} + (h_\varepsilon)_{\alpha\bar{s}}dz^\alpha\wedge d\bar{s} + (h_\varepsilon)_{\alpha\bar{\beta}}dz^\alpha\wedge dz^{\bar{\beta}}\right).$$

For each $y\in\mathbf{D}$, $(h_\varepsilon)_{\alpha\bar{\beta}}(\cdot,y)$ gives a Kähler metric on X_y. (If there is no confusion, we simply write $(h_\varepsilon)_{\alpha\bar{\beta}}$.) Thus we can define contraction and covariant derivative on each X_y with respect to $(h_\varepsilon)_{\alpha\bar{\beta}}$. We use raising and lowering of indices as well as the semi-colon for the contractions and the covariant derivatives with respect to the Kähler metric $(h_\varepsilon)_{\alpha\bar{\beta}}$, respectively, on the fiber X_y. We denote by $\Delta_{\rho_\varepsilon} = \Delta_{\rho_\varepsilon|_{X_y}}$ the Laplace-Beltrami operator with negative eigenvalues on the fiber X_y with respect to $\rho_\varepsilon|_{X_y}$.

By raising of indices, we can write the horizontal lift v_{ρ_ε} of $v = \partial/\partial s$ with respect to ρ_ε by

$$v_{\rho_\varepsilon} = \frac{\partial}{\partial s} - (h_\varepsilon)_{s\bar{\beta}}(h_\varepsilon)^{\bar{\beta}\alpha}\frac{\partial}{\partial z^\alpha} = \frac{\partial}{\partial s} - {(h_\varepsilon)_s}^{\alpha}\frac{\partial}{\partial z^\alpha}.$$

Then $\bar{\partial}v_{\rho_\varepsilon}$ is a $T'X_y$-valued $(0,1)$-form which is defined by

$$\begin{aligned}\bar{\partial}v_{\rho_\varepsilon} &= \bar{\partial}\left(\frac{\partial}{\partial s} - {(h_\varepsilon)_s}^{\alpha}\frac{\partial}{\partial z^\alpha}\right)\\ &= \left(-\bar{\partial}{(h_\varepsilon)_s}^{\alpha}\right)\otimes\frac{\partial}{\partial z^\alpha}\\ &= -\frac{\partial {(h_\varepsilon)_s}^{\alpha}}{\partial z^{\bar{\beta}}}dz^{\bar{\beta}}\otimes\frac{\partial}{\partial z^\alpha}.\end{aligned}$$

Since $(h_\varepsilon)_{\alpha\bar{\beta}}$ is a Kähler metric and we use holomorphic coordinates, $\bar{\partial}v_{\rho_\varepsilon}$ is written by

$$\bar{\partial}v_{\rho_\varepsilon} = -{(h_\varepsilon)_s}^{\alpha}{}_{;\bar{\beta}}dz^{\bar{\beta}}\otimes\frac{\partial}{\partial z^\alpha}.$$

Then Remark 2.2 says that the geodesic curvature $c(\rho_\varepsilon) : X \to \mathbb{R}$ is given by

$$\begin{aligned} c(\rho_\varepsilon)(z,s) &= \langle v_{\rho_\varepsilon}, v_{\rho_\varepsilon} \rangle_{\rho_\varepsilon} \\ &= (h_\varepsilon)_{s\bar{s}} - (h_\varepsilon)_{s\bar{\beta}} (h_\varepsilon)^{\bar{\beta}\alpha} (h_\varepsilon)_{\alpha\bar{s}}. \end{aligned}$$

The following theorem is inspired by Schumacher's method in [30]. Păun generalized the computation to the twisted Kähler-Einstein metric case ([28]). (See also [9].)

Theorem 5.1. *The following partial differential equation holds on each fiber X_y:*

$$-\Delta_{\rho_\varepsilon} c(\rho_\varepsilon) + \varepsilon c(\rho_\varepsilon) = \varepsilon \omega(v_{\rho_\varepsilon}, \overline{v_{\rho_\varepsilon}}) + \left|\bar{\partial} v_{\rho_\varepsilon}\right|^2_{\rho_\varepsilon} - \Theta_{s\bar{s}}(E),$$

where $\left|\bar{\partial} v_{\rho_\varepsilon}\right|_{\rho_\varepsilon}$ is the pointwise norm of $\bar{\partial} v_{\rho_\varepsilon}$ with respect to the Kähler metric $\rho_\varepsilon|_{X_y}$.

Proof. We fix a fiber X_y and $\varepsilon > 0$. During this proof, if there is no confusion, we omit the subscript ε in the components in ρ_ε for simplicity, namely, we write as follows:

$$h_{s\bar{s}} = (h_\varepsilon)_{s\bar{s}}, \quad h_{s\bar{\beta}} = (h_\varepsilon)_{s\bar{\beta}} \quad \text{and} \quad h_{\alpha\bar{\beta}} = (h_\varepsilon)_{\alpha\bar{\beta}}.$$

We have to compute the following:

$$\Delta_{\rho_\varepsilon} c(\rho_\varepsilon) = h^{\bar{\delta}\gamma} (c(\rho_\varepsilon))_{;\gamma\bar{\delta}} = h^{\bar{\delta}\gamma} \left(h_{s\bar{s}} - h_{s\bar{\beta}} h^{\bar{\beta}\alpha} h_{\alpha\bar{s}} \right)_{;\gamma\bar{\delta}}.$$

First we consider the term $h^{\bar{\delta}\gamma} h_{s\bar{s};\gamma\bar{\delta}}$. Since ω is a Kähler form on X, ρ_ε is locally $\partial\bar{\partial}$-exact. So we have that

$$h_{s\bar{s};\gamma\bar{\delta}} = \frac{\partial^2 h_{s\bar{s}}}{\partial z^\gamma \partial z^{\bar{\delta}}} = \frac{\partial^2}{\partial s \partial \bar{s}} h_{\gamma\bar{\delta}}.$$

Then it follows that

$$\begin{aligned} h^{\bar{\delta}\gamma} h_{s\bar{s};\gamma\bar{\delta}} &= h^{\bar{\delta}\gamma} \frac{\partial^2}{\partial s \partial \bar{s}} h_{\gamma\bar{\delta}} \\ &= \frac{\partial}{\partial s} \left(h^{\bar{\delta}\gamma} \frac{\partial}{\partial \bar{s}} h_{\gamma\bar{\delta}} \right) - \frac{\partial h^{\bar{\delta}\gamma}}{\partial s} \frac{\partial h_{\gamma\bar{\delta}}}{\partial \bar{s}} \\ &= \frac{\partial^2}{\partial s \partial \bar{s}} \log\det(h_{\alpha\bar{\beta}}) + h^{\bar{\delta}\alpha} \frac{\partial h_{\alpha\bar{\beta}}}{\partial s} h^{\bar{\beta}\gamma} \frac{\partial h_{\gamma\bar{\delta}}}{\partial \bar{s}} \end{aligned}$$

By (5.3), we have

$$\frac{\partial^2}{\partial s \partial \bar{s}} \log\det(h_{\alpha\bar{\beta}}) = \varepsilon \rho_\varepsilon \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \bar{s}} \right) - \varepsilon \omega \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial \bar{s}} \right) + \Theta_{s\bar{s}}(E).$$

Hence it follows that

$$h^{\bar{\delta}\gamma} h_{s\bar{s};\gamma\bar{\delta}} = \varepsilon \left(h_{s\bar{s}} - g_{s\bar{s}} \right) + \Theta_{s\bar{s}}(E) + h_{s\bar{\beta};\alpha} h_{\bar{s}\gamma;\bar{\delta}} h^{\bar{\beta}\gamma} h^{\bar{\delta}\alpha}. \tag{5.4}$$

Next we consider the term $h^{\bar{\delta}\gamma} \left(h_{s\bar{\beta}} h^{\bar{\beta}\alpha} h_{\alpha\bar{s}} \right)_{;\gamma\bar{\delta}}$, which can be written by

$$h^{\bar{\delta}\gamma} \left(h_s{}^\alpha h_{\alpha\bar{s}} \right)_{;\gamma\bar{\delta}}.$$

Define a tensor $\{A_s{}^\alpha{}_{\bar{\beta}}\}$ by

$$A_s{}^\alpha{}_{\bar{\beta}} = -h_s{}^\alpha{}_{;\bar{\beta}}.$$

Then it follows that

$$\bar{\partial} v_\rho = A_s{}^\alpha{}_{\bar{\beta}} \frac{\partial}{\partial z^\alpha} \otimes dz^{\bar{\beta}}.$$

Hence we have

$$
\begin{aligned}
h^{\bar{\delta}\gamma}\left(h_s{}^{\sigma}h_{\bar{s}\delta}\right)_{;\gamma\bar{\delta}} &= h^{\bar{\delta}\gamma}\left(h_s{}^{\sigma}{}_{;\gamma\bar{\delta}}h_{\bar{s}\sigma} + A_s{}^{\sigma}{}_{\bar{\delta}}A_{\bar{s}\sigma\gamma} + h_s{}^{\sigma}{}_{;\bar{\delta}}h_{\bar{s}\sigma;\bar{\delta}} + h_s{}^{\sigma}A_{\bar{s}\sigma\gamma;\bar{\delta}}\right) \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

First of all, it is obvious that

$$
I_2 = A_s{}^{\sigma}{}_{\bar{\delta}}A_{\bar{s}\sigma\gamma}h^{\bar{\delta}\gamma} = \left|\bar{\partial}v_{\rho_\varepsilon}\right|^2_{\rho_\varepsilon}.
$$

And the term I_3 is equal to $h_{s\bar{\beta};\alpha}h_{\bar{s}\gamma;\bar{\delta}}h^{\bar{\beta}\gamma}h^{\bar{\delta}\alpha}$, which is appeared in (5.4). So these terms are cancelled in the last computation.

Before computing I_1 and I_4, we introduce some ingredients. Let $R^{\delta}{}_{\alpha\bar{\beta}\gamma}$ be a Riemann curvature tensor of $\rho_\varepsilon|_{X_y}$. Then by the commutation formula for covariants derivatives, we have

$$
T^{\alpha}{}_{;\bar{\beta}\gamma} - T^{\alpha}{}_{;\gamma\bar{\beta}} = R^{\alpha}{}_{\delta\bar{\beta}\gamma}T^{\delta}. \tag{5.5}
$$

Let $R_{\alpha\bar{\beta}} := R^{\gamma}{}_{\alpha\bar{\beta}\gamma}$ be the Ricci tensor of $\rho_\varepsilon|_{X_y}$. By the definition of $h^{\rho_\varepsilon}_{X/Y}$ in Remark 2.3, we have

$$
\Theta_{h^{\rho_\varepsilon}_{X/Y}}|_{X_y} = -\mathrm{Ric}(\rho_\varepsilon|_{X_y}).
$$

Hence it follows from (5.3) that

$$
R_{\alpha\bar{\beta}} = \varepsilon h_{\alpha\bar{\beta}} - \varepsilon g_{\alpha\bar{\beta}}.
$$

Lemma 5.2. *Let $\bar{\partial}^*_{\rho_\varepsilon}$ be the adjoint of $\bar{\partial}$ with respect to the L^2-inner product with $\rho_\varepsilon|_{X_y}$, which is defined by*

$$
\bar{\partial}^*\left(A_s{}^{\alpha}{}_{\bar{\beta}}\frac{\partial}{\partial z^\alpha}\otimes dz^{\bar{\beta}}\right) := h^{\bar{\beta}\gamma}A_s{}^{\alpha}{}_{\bar{\beta};\gamma}\frac{\partial}{\partial z^\alpha}
$$

Then we have the following:

$$
\bar{\partial}^*\left(\bar{\partial}v_{\rho^\varepsilon}\right) = \varepsilon\left(g_{s\bar{\delta}}h^{\bar{\delta}\alpha} - h_{s\bar{\delta}}g^{\bar{\delta}\alpha}\right)\frac{\partial}{\partial z^\alpha}. \tag{5.6}
$$

In particular, we have

$$
h^{\bar{\beta}\gamma}A_s{}^{\alpha}{}_{\bar{\beta};\gamma} = \varepsilon\left(g_{s\bar{\delta}}h^{\bar{\delta}\alpha} - h_{s\bar{\delta}}g^{\bar{\delta}\alpha}\right).
$$

Proof. Since the Riemannian connection induced by a Kähler metric is torsion-free, we have

$$
h^{\bar{\beta}\gamma}A_s{}^{\alpha}{}_{\bar{\beta};\gamma} = -h^{\bar{\beta}\gamma}h^{\bar{\delta}\alpha}h_{s\bar{\delta};\bar{\beta}\gamma} = -h^{\bar{\beta}\gamma}h^{\bar{\delta}\alpha}h_{s\bar{\beta};\bar{\delta}\gamma}.
$$

By (5.3) and (5.5), it follows that

$$\begin{aligned}
h^{\bar\beta\gamma}A_s{}^{\alpha}{}_{\bar\beta;\gamma} &= -h^{\bar\beta\gamma}h^{\bar\delta\alpha}\left[h_{s\bar\beta;\gamma\bar\delta} - h_{s\bar\tau}R^{\bar\tau}{}_{\bar\beta\bar\delta\gamma}\right]\\
&= -h^{\bar\delta\alpha}\left[\left(h^{\bar\beta\gamma}\frac{\partial h_{\bar\beta\gamma}}{\partial s}\right)_{;\bar\delta} - h_{s\bar\tau}h^{\bar\beta\gamma}R^{\bar\tau}{}_{\bar\beta\bar\delta\gamma}\right]\\
&= -h^{\bar\delta\alpha}\left[\left(\frac{\partial}{\partial s}\log\det(h_{\alpha\bar\beta})\right)_{;\bar\delta} + h_{s\bar\tau}R^{\bar\tau}{}_{\bar\delta}\right]\\
&= -h^{\bar\delta\alpha}\left[(\Theta_{h^{\rho_\varepsilon}_{X/Y}})_{s\bar\delta} + h_{s\bar\tau}h^{\bar\tau\gamma}R_{\gamma\bar\delta}\right]\\
&= -h^{\bar\delta\alpha}\left[(\Theta_{h^{\rho_\varepsilon}_{X/Y}})_{s\bar\delta} - h_{s\bar\tau}h^{\bar\tau\gamma}(\Theta_{h^{\rho_\varepsilon}_{X/Y}})_{\gamma\bar\delta}\right]\\
&= -\varepsilon h^{\bar\delta\alpha}\left[h_{s\bar\delta} - g_{s\bar\delta} - h_{s\bar\tau}h^{\bar\tau\gamma}\left(h_{\gamma\bar\delta} - g_{\gamma\bar\delta}\right)\right]\\
&= \varepsilon\left(g_{s\bar\delta}h^{\bar\delta\alpha} - h_{s\bar\delta}g^{\bar\delta\alpha}\right)
\end{aligned}$$

This completes the proof. □

Next we compute the term I_1:

$$\begin{aligned}
I_1 &= h_{\bar s\sigma}h_s{}^{\sigma}{}_{;\gamma\bar\delta}h^{\bar\delta\gamma}\\
&= h_{\bar s\sigma}\left(-A_s{}^{\sigma}{}_{\bar\delta;\gamma}h^{\bar\delta\gamma} + h_s{}^{\lambda}R^{\sigma}{}_{\lambda\gamma\bar\delta}h^{\bar\delta\gamma}\right)\\
&= h_{\bar s\sigma}\left[-\varepsilon\left(g_{s\bar\delta}h^{\bar\delta\sigma} - h_{s\bar\delta}g^{\bar\delta\sigma}\right) - h_s{}^{\lambda}R^{\sigma}{}_{\lambda}\right]\\
&= h_{\bar s\sigma}\left[-\varepsilon\left(g_{s\bar\delta}h^{\bar\delta\sigma} - h_{s\bar\delta}g^{\bar\delta\sigma}\right) - h_{s\bar\lambda}R^{\sigma\bar\lambda}\right]\\
&= h_{\bar s\sigma}\left[-\varepsilon\left(g_{s\bar\delta}h^{\bar\delta\sigma} - h_{s\bar\delta}g^{\bar\delta\sigma}\right) + h_{s\bar\lambda}\varepsilon\left(h^{\sigma\bar\lambda} - g^{\sigma\bar\lambda}\right)\right]\\
&= \varepsilon h_{\bar s\sigma}\left[-g_{s\bar\delta}h^{\bar\delta\sigma} + h_{s\bar\delta}g^{\bar\delta\sigma} + h_{s\bar\lambda}\left(h^{\sigma\bar\lambda} - g^{\sigma\bar\lambda}\right)\right]\\
&= \varepsilon\left(h_{s\bar\beta}h^{\bar\beta\alpha}h_{\alpha\bar s} - g_{s\bar\beta}h^{\bar\beta\alpha}h_{\alpha\bar s}\right).
\end{aligned}$$

Finally we compute the term I_4:

$$\begin{aligned}
I_4 &= h^{\gamma\bar\delta}h_s{}^{\sigma}A_{\bar s\sigma\gamma;\bar\delta}\\
&= h_{s\bar\sigma}h^{\gamma\bar\delta}A_{\bar s}{}^{\bar\sigma}{}_{\gamma;\bar\delta}\\
&= h_{s\bar\sigma}\varepsilon\left(g_{\bar s\delta}h^{\delta\bar\sigma} - h_{\bar s\delta}g^{\delta\bar\sigma}\right)\\
&= \varepsilon\left(h_{s\bar\beta}h^{\bar\beta\alpha}g_{\alpha\bar s} - h_{s\bar\beta}g^{\bar\beta\alpha}h_{\alpha\bar s}\right).
\end{aligned}$$

Together with all computations, it follows that

$$\begin{aligned}\Delta_{\rho_\varepsilon} c(\rho_\varepsilon) &= \varepsilon(h_{s\bar{s}} - g_{s\bar{s}}) + \Theta_{s\bar{s}}(E) - \left|\bar{\partial} v_{\rho_\varepsilon}\right|^2_{\rho_\varepsilon} \\ &\quad - \varepsilon\left(h_{s\bar{\beta}} h^{\bar{\beta}\alpha} h_{\alpha\bar{s}} - g_{s\bar{\beta}} h^{\bar{\beta}\alpha} h_{\alpha\bar{s}}\right) \\ &\quad - \varepsilon\left(h_{s\bar{\beta}} h^{\bar{\beta}\alpha} g_{\alpha\bar{s}} - h_{s\bar{\beta}} g^{\bar{\beta}\alpha} h_{\alpha\bar{s}}\right) \\ &= \Theta_{s\bar{s}}(E) - \left|\bar{\partial} v_{\rho_\varepsilon}\right|^2_{\rho_\varepsilon} + \varepsilon\left(h_{s\bar{s}} - h_{s\bar{\beta}} h^{\bar{\beta}\alpha} h_{\alpha\bar{s}}\right) \\ &\quad + \varepsilon\left(g_{s\bar{s}} - g_{s\bar{\beta}} h^{\bar{\beta}\alpha} h_{\alpha\bar{s}} - h_{s\bar{\beta}} h^{\bar{\beta}\alpha} g_{\alpha\bar{s}} + h_{s\bar{\beta}} g^{\bar{\beta}\alpha} h_{\alpha\bar{s}}\right).\end{aligned}$$

Since

$$\omega(v_{\rho_\varepsilon}, \overline{v_{\rho_\varepsilon}}) = g_{s\bar{s}} - g_{s\bar{\beta}} h^{\bar{\beta}\alpha} h_{\alpha\bar{s}} - h_{s\bar{\beta}} h^{\bar{\beta}\alpha} g_{\alpha\bar{s}} + h_{s\bar{\beta}} g^{\bar{\beta}\alpha} h_{\alpha\bar{s}},$$

it follows that

$$-\Delta_{\rho_\varepsilon} c(\rho_\varepsilon) + \varepsilon c(\rho_\varepsilon) = \varepsilon\omega(v_{\rho_\varepsilon}, \overline{v_{\rho_\varepsilon}}) + \left|\bar{\partial} v_{\rho_\varepsilon}\right|^2_{\rho_\varepsilon} - \Theta_{s\bar{s}}(E).$$

Therefore, we have the conclusion. □

Corollary 5.3. *Let ρ be the fiberwise Ricci-flat metric in Theorem 1.1. Then the following PDE holds on each fiber X_y:*

$$-\Delta_\rho c(\rho) = \left|\bar{\partial} v_\rho\right|^2_\rho - \Theta_{s\bar{s}}(E).$$

Proof. Recall that the fiberwise Ricci-flat metric ρ satisfies the following:

$$\Theta_{h^\rho_{X/Y}}(K_{X/Y}) = -dd^c \log \|u\|^2_s = \Theta(E)$$

If we apply the same computation with the proof of Theorem 5.1 to ρ using the above equation, then we have the conclusion.

On the other hand, it is also an easy consequence of the convergence of the form ρ_ε to ρ as $\varepsilon \to 0$ by passing through a subsequence for each $y \in Y$. (More precisely, the function φ_ε converges to φ as $\varepsilon \to 0$.) This will be proved in the next section. □

Remark 5.4. The computations in Corollary 5.3 do not use the normalization condition of φ. Hence it is easy to see that for any d-closed smooth real $(1,1)$-form τ whose restriction on each fiber is the Ricci-flat metric we have

$$-\Delta_\tau c(\tau) = \left|\bar{\partial} v_\tau\right|^2_\tau - \Theta_{s\bar{s}}(E).$$

Now we are at the position of proving the positivity of the direct image $p_*\rho^{n+1}$. As we mentioned in Subsection 2.1, it is enough to show that the fiber integral $\int_{X_s} c(\rho)\rho^n$ is positive. It follows from Theorem 5.1 and Proposition 4.6 that

$$\begin{aligned}\int_{X/\mathbf{D}} c(\rho_\varepsilon)\rho_\varepsilon^n &= \int_{X_s} \frac{1}{\varepsilon}\left[\Delta_{\rho_\varepsilon} c(\rho_\varepsilon) + \left|\bar{\partial} v_{\rho_\varepsilon}\right|^2_{\rho_\varepsilon} - \Theta_{s\bar{s}}(E) + \varepsilon\omega(v_{\rho_\varepsilon}, \bar{v}_{\rho_\varepsilon})\right]\rho_\varepsilon^n \\ &= \frac{1}{\varepsilon}\left[\left\|\bar{\partial} v_{\rho_\varepsilon}\right\|^2_{L^2_{\rho_\varepsilon}(X_s)} - \Theta_{s\bar{s}}(E)\right] + \int_{X_s} \omega(v_{\rho_\varepsilon}, \bar{v}_{\rho_\varepsilon})\rho_\varepsilon^n \\ &\geq \int_{X_s} \omega(v_{\rho_\varepsilon}, \bar{v}_{\rho_\varepsilon})\rho_\varepsilon^n.\end{aligned}$$

We already know that on each fiber X_s, the $\rho_\varepsilon|_{X_s}$ converges to $\rho|_{X_s}$ by Corollary 3.5. Therefore, Proposition 5.5, which will be proved in the next section, says that

$$\int_{X/\mathbf{D}} c(\rho)\rho^n \geq \int_{X_s} \omega(v_\rho, \bar{v}_\rho)\rho^n. \tag{5.7}$$

In paticular, $p_*\rho^{n+1}$ is positive.

Proposition 5.5. *On each fiber X_y, there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ converging to 0 as $j \to \infty$ such that*

$$c(\rho_{\varepsilon_j}) \to c(\rho) \quad \textit{and} \quad \bar{\partial} v_{\rho_{\varepsilon_j}} \to \bar{\partial} v_\rho \quad \textit{as} \quad j \to \infty.$$

We end this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. It is enough to prove when the family is over the unit disc $\mathbf{D}$ in $\mathbb{C}$. Let $s \in \mathbf{D}$. Recall that $c(\rho)$ satisfies

$$-\Delta_\rho c(\rho) = \left|\bar{\partial} v_\rho\right|_\rho^2 - \Theta_{s\bar{s}}(E).$$

It follows from the Green kernel formula that

$$c(\rho) = \int_{X_s} c(\rho)\rho^n + \int_{X_s} G_s(z,w)\left(\left|\bar{\partial} v_\rho\right|_\rho^2 - \Theta_{s\bar{s}}(E)\right)\rho^n$$

Since the integral of $G_s(z,w)$ is zero, it follows that

$$\begin{aligned} c(\rho) &\geq \int_{X_s} c(\rho)\rho^n - \int_{X_s} G_s(z,w)\left|\bar{\partial} v_\rho\right|_\rho^2 \rho^n \\ &= \int_{X_s} c(\rho)\rho^n - K(s)\int_{X_s}\left|\bar{\partial} v_\rho\right|_\rho^2 \rho^n. \\ &= \int_{X_s} c(\rho)\rho^n - K(s)\omega^{WP}(v,\bar{v}). \end{aligned}$$

Equation (5.7) implies that

$$c(\rho) + K(s)\omega^{WP}(v,\bar{v}) \geq \int_{X_s} c(\rho)\rho^n \geq \int_{X_s} \omega(v_\rho,\bar{v}_\rho)\rho^n > 0.$$

On the other hand, it is easy to see that

$$c\left(\rho + K(s)\omega^{WP}\right) = c(\rho) + K(s)\omega^{WP}(v,\bar{v}).$$

This concludes that $\rho + K(s)\omega^{WP}$ is positive on X. □

6. Approximation of the geodesic curvature

In this section, we shall prove Proposition 5.5.

First we recall the setting: Let $p : X \to \mathbf{D}$ be a Calabi-Yau fibration and let ω be a fixed Kähler form on X. For each fiber X_y, we have a unique solution $\varphi_{y,\varepsilon}$ of the following complex Monge-Ampère equation:

$$\begin{aligned} &(\omega_y + dd^c\varphi_{y,\varepsilon})^n = e^{\varepsilon\varphi_{y,\varepsilon}} e^{\eta_y}(\omega_y)^n \quad \text{and} \\ &\omega_y + dd^c\varphi_{y,\varepsilon} > 0, \end{aligned} \tag{6.1}$$

where η is defined in Section 4. As we mentioned, we can consider φ_ε as a smooth function on X by letting

$$\varphi_\varepsilon(x) := \varphi_{y,\varepsilon}(x),$$

where $y = p(x)$. Denote by $\rho_\varepsilon = \omega + dd^c\varphi_\varepsilon$.

On the other hand, for each fiber X_y, we have the solution φ_y of the following complex Monge-Ampère equation:

$$
\begin{aligned}
(\omega_y + dd^c\varphi_y)^n &= e^{\eta|_{X_y}}(\omega_y)^n, \\
\omega_y + dd^c\varphi_y &> 0,
\end{aligned} \tag{6.2}
$$

which is normalized by

$$\int_{X_y} \varphi_y e^{\eta_y}(\omega_y)^n = 0. \tag{6.3}$$

Then φ is a smooth function on X. We denote by $\rho = \omega + dd^c\varphi$. It is remarkable to note that ρ_ε and ρ are uniformly equivalent on X_y by Proposition 3.2.

In this section, we write the horizontal lifting v_ρ of $\partial/\partial s$ with respect to ρ as follows:

$$v_\rho = \frac{\partial}{\partial s} + a_s{}^\alpha \frac{\partial}{\partial z^\gamma} = \frac{\partial}{\partial s} - h_{s\bar\beta} h^{\bar\beta\alpha} \frac{\partial}{\partial z^\gamma}.$$

in an admissible coordinate (z, s) in X.

Theorem 6.1. *For a fixed fiber X_y, the following holds:*

$$\varphi_\varepsilon \to \varphi, \quad v_\rho\varphi_\varepsilon \to v_\rho\varphi \quad \text{and} \quad \overline{v_\rho}v_\rho\varphi_\varepsilon \to \overline{v_\rho}v_\rho\varphi$$

as $\varepsilon \to 0$ in $C^{k,\alpha}(X_y)$-topology for any $k \in \mathbb{N}$ and $\alpha \in (0,1)$ by passing through a subsequence.

It is obvious that this theorem implies Proposition 5.5.

In the proof, we fix a fiber X_y and omit the subscript y, if there is no confusion. Every convergence means the convergence by passing through a subsequence in the topology of $C^{k,\alpha}(X_y)$ for any $k \in \mathbb{N}$ and $\alpha \in (0,1)$.

It is easy to see that Corollary 3.5 yields the first assertion. This also implies that there exists a uniform constant $C > 0$ such that

$$\frac{1}{C}\omega_y < \rho_\varepsilon|_{X_y} < C\omega_y, \tag{6.4}$$

for $0 < \varepsilon \leq 1$.

Before going to the further proof of Theorem 6.1, we introduce the following proposition about the fiber integral.

Proposition 6.2. *Let τ be a d-closed real $(1,1)$-form on X whose restriction on each fiber X_s is positive definite. For a smooth function f on X, we have*

$$\frac{\partial}{\partial s}\int_{X_s} f\tau^n = \int_{X_s} L_{v_\tau}(f\tau^n) = \int_{X_s} (v_\tau f)\tau^n.$$

In particular, if $\int_{X_s} f\tau^n = 0$ for $s \in \mathbf{D}$, then

$$\int_{X_s} (v_\tau f)\tau^n = 0.$$

Proof. The first equality is mentioned in Section 3.2. Cartan's magic formula and Stokes' theorem imply that

$$\begin{aligned}\frac{\partial}{\partial s}\int_{X_s} f\tau^n &= \int_{X_s} L_{v_\tau}\left(f\tau^n\right)\\ &= \int_{X_s}\left(d\circ i_{v_\tau} + i_{v_\tau}\circ d\right)\left(f\tau^n\right)\\ &= \int_{X_s} d\left(i_{v_\tau}\left(f\tau^n\right)\right) + \int_{X_s} i_{v_\tau}\left(df\wedge\tau^n\right)\\ &= \int_{X_s}(v_\tau f)\tau^n - \int_{X_s} df\wedge i_{v_\tau}(\tau^n).\end{aligned}$$

On the other hand, Lemma 2.4 implies that

$$i_{v_\tau}(\tau^n) = i_{v_\tau}(\tau)\wedge\tau^{n-1} = \sqrt{-1}c(\tau)\wedge\tau^{n-1}\wedge d\bar{s}.$$

Hence we have

$$\int_{X_s} df\wedge i_{v_\tau}(\tau^n) = \int_{X_s}\sqrt{-1}c(\tau)df\wedge\tau^{n-1}\wedge d\bar{s} = 0.$$

This completes the proof. □

Now we go back to the proof of the second assertion. Taking logarithm of (6.1) and differentiating it with respect to v_ρ, we have

$$(h_\varepsilon)^{\bar{\beta}\alpha} v_\rho\left(g_{\alpha\bar{\beta}} + (\varphi_\varepsilon)_{\alpha\bar{\beta}}\right) = \varepsilon v_\rho\varphi_\varepsilon + v_\rho\eta + g^{\bar{\beta}\alpha}v_\rho(g_{\alpha\bar{\beta}}).$$

As in Section 3, we have

$$-\Delta_{\rho_\varepsilon}\left(v_\rho\varphi_\varepsilon\right) + \varepsilon\left(v_\rho\varphi_\varepsilon\right) = -v_\rho\eta + (h_\varepsilon)^{\alpha\bar{\beta}}\left(v_\rho\left(g_{\alpha\bar{\beta}}\right) + [v_\rho,\varphi_\varepsilon]_{\alpha\bar{\beta}}\right) - g^{\alpha\bar{\beta}}v_\rho\left(g_{\alpha\bar{\beta}}\right),$$

where $\Delta_{\rho_\varepsilon}$ is the Laplace-Beltrami operator of ρ_ε and

$$\begin{aligned}[v_\rho,\varphi_\varepsilon]_{\alpha\bar{\beta}} &= v_\rho((\varphi_\varepsilon)_{\alpha\bar{\beta}}) - (v_\rho(\varphi_\varepsilon))_{\alpha\bar{\beta}}\\ &= -a_s{}^{\gamma}{}_{\alpha\bar{\beta}}(\varphi_\varepsilon)_\gamma - a_s{}^{\gamma}{}_{\alpha}(\varphi_\varepsilon)_{\gamma\bar{\beta}} - a_s{}^{\gamma}{}_{\bar{\beta}}(\varphi_\varepsilon)_{\alpha\gamma}.\end{aligned}$$

We denote the right hand side by R_ε. Hence $v_\rho\varphi_\varepsilon$ satisfies the following equation:

$$-\Delta_{\rho_\varepsilon}(v_\rho\varphi_\varepsilon) + \varepsilon(v_\rho\varphi_\varepsilon) = R_\varepsilon. \tag{6.5}$$

Then Proposition 3.6 implies that there exists a uniform constant $C > 0$ such that

$$\|v_\rho\varphi_\varepsilon\|_{C^{k,\alpha}(X_s)} < C.$$

By the same computation to (6.2), $v_\rho\varphi$ satisfies that

$$-\Delta_\rho(v_\rho\varphi) = R, \tag{6.6}$$

where

$$R = -v_\rho\eta + h^{\alpha\bar{\beta}}\left(v_\rho\left(g_{\alpha\bar{\beta}}\right) + [v_\rho,\varphi]_{\alpha\bar{\beta}}\right) - g^{\alpha\bar{\beta}}v_\rho\left(g_{\alpha\bar{\beta}}\right).$$

Since φ_ε converges to φ and $[v_\rho,\varphi_\varepsilon]_{\alpha\bar{\beta}}$ does not include s-derivative of φ_ε, we have

$$(h_\varepsilon)^{\bar{\beta}\alpha}\to h^{\bar{\beta}\alpha} \quad\text{and}\quad [v_\rho,\varphi_\varepsilon]_{\alpha\bar{\beta}}\to[v_\rho,\varphi]_{\alpha\bar{\beta}} \quad\text{as}\quad \varepsilon\to 0.$$

It follows that Equation (6.5) converges to Equation (6.6) as $\varepsilon\to 0$. Since Proposition 6.2 says that $v_\rho\varphi$ is the unique solution of (6.6) which satisfies that

$$\int_{X_s}(v_\rho\varphi)\rho^n = 0,$$

the following Lemma completes the proof.

Lemma 6.3. *The following holds:*

$$\lim_{\varepsilon\to 0}\int_{X_s}(v_\rho\varphi_\varepsilon)\rho^n = 0.$$

Proof. Integrating (6.1), we have

$$1 = \int_{X_s} e^{\varepsilon\varphi_\varepsilon+\eta}\omega^n.$$

Differentiating with respect to s, we have

$$0 = \frac{\partial}{\partial s}\int_{X_s} e^{\varepsilon\varphi_\varepsilon+\eta}\omega^n = \int_{X_s} v_\rho(e^{\varepsilon\varphi_\varepsilon})\rho^n = \varepsilon\int_{X_s}(v_\rho\varphi_\varepsilon)e^{\varepsilon\varphi_\varepsilon}\rho^n.$$

Since $e^{\varepsilon\varphi_\varepsilon}(\rho_\varepsilon)^n = \rho^n$ on each fiber X_s,

$$\int_{X_s}(v_\rho\varphi_\varepsilon)(\rho_\varepsilon)^n = 0.$$

Since ρ_ε and ρ is uniformly equivalent on X_s, this completes the proof. □

It remains only to prove the last assertion.

Differentiating (6.5) with respect to $\overline{v_\rho}$, we have

$$\begin{aligned}-\Delta_{\rho_\varepsilon}(\overline{v_\rho}v_\rho\varphi_\varepsilon) + \varepsilon(\overline{v_\rho}v_\rho\varphi_\varepsilon) =& \overline{v_\rho}\left((h^\varepsilon)^{\bar\beta\alpha}\right)\cdot(v_\rho(\varphi_\varepsilon))_{\alpha\bar\beta} + \overline{v_\rho}(R_\varepsilon)\\ &+ (h_\varepsilon)^{\bar\beta\alpha}[\overline{v_\rho}, v_\rho\varphi_\varepsilon]_{\alpha\bar\beta}.\end{aligned} \tag{6.7}$$

Then Proposition 3.6 implies that there exists a uniform constant $C>0$ such that

$$\|\overline{v_\rho}v_\rho\varphi_\varepsilon\|_{C^{k,\alpha}(X_s)} < C.$$

By the same way, $\overline{v_\rho}v_\rho\varphi$ satisfies that

$$-\Delta_\rho\overline{v_\rho}v_\rho\varphi = \overline{v_\rho}\left(h^{\bar\beta\alpha}\right)\cdot(v_\rho\varphi)_{\alpha\bar\beta} + \overline{v_\rho}R + h^{\bar\beta\alpha}[\overline{v_\rho}, v_\rho\varphi]_{\alpha\bar\beta}. \tag{6.8}$$

We already know that $\varphi_\varepsilon\to\varphi$ and $v_\rho\varphi_\varepsilon\to v_\rho\varphi$ as $\varepsilon\to 0$ on X_y. Hence the similar argument says that the RHS of (6.7) converges to the RHS of (6.8) as $\varepsilon\to 0$. Since Proposition 6.2 says that $\overline{v_\rho}v_\rho\varphi$ is the unique solution of (6.8) which satisfies that

$$\int_{X_s}(\overline{v_\rho}v_\rho\varphi)\rho^n = 0,$$

As the previous argument, the following lemma completes the proof.

Lemma 6.4. *The following holds:*

$$\lim_{\varepsilon\to 0}\int_{X_s}(\overline{v_\rho}v_\rho\varphi_\varepsilon)\rho^n = 0.$$

Proof. Integrating (6.1), we have

$$1 = \int_{X_s} e^{\varepsilon\varphi_\varepsilon+\eta}\omega^n.$$

Differentiating with respect to s and $\bar{s}$, we have

$$
\begin{aligned}
0 = \frac{\partial^2}{\partial \bar{s} \partial s} \int_{X_s} e^{\varepsilon\varphi_\varepsilon + \eta} \omega^n &= \int_{X_s} (\overline{v_\rho} v_\rho e^{\varepsilon\varphi_\varepsilon}) \rho^n \\
&= \varepsilon \int_{X_s} (\overline{v_\rho} v_\rho \varphi_\varepsilon) e^{\varepsilon\varphi_\varepsilon} \rho^n + \varepsilon^2 \int_{X_s} |v_\rho \varphi_\varepsilon|^2 e^{\varepsilon\varphi_\varepsilon} \rho^n.
\end{aligned}
$$

Since φ_ε and $v_\rho\varphi_\varepsilon$ are uniformly bounded, it follows that

$$
\int_{X_s} (\overline{v_\rho} v_\rho \varphi_\varepsilon)(\rho_\varepsilon)^n \to 0 \quad \text{as} \quad \varepsilon \to 0.
$$

This completes the proof as in the proof of Lemma 6.3. □

7. Some remarks

As we mentioned in Introduction, our method does not show the positivity or semipositivity of the fiberwise Ricci-flat metrics. But in a special case, we have the positivity. In the next section, we will introduce an example which is in this case.

Corollary 7.1. *Suppose that $\left|\bar{\partial} v_\rho\right|_\rho$ depends only on s-variable (, i.e., it is a constant on each fiber.) Then ρ is positive on X.*

Proof. Since $\left|\bar{\partial} v_\rho\right|_\rho$ is constant on each fiber X_s, Proposition 4.5 says that $\left|\bar{\partial} v_\rho\right|_\rho^2 = \Theta_{s\bar{s}}(E)$. It follows that

$$
-\Delta_\rho c(\rho) = 0,
$$

i.e., $c(\rho)$ is a constant on each fiber. Hence we have

$$
c(\rho) = \int_{X_s} c(\rho)\rho^n.
$$

Then Theorem 1.1 completes the proof. □

Now we consider a different type of fiberwise Ricci-flat metric. Let $p : X \to \mathbf{D}$ be a Calabi-Yau fibration and let ω be a fixed Kähler form on X. By the same argument, there exists a unique smooth function ψ in X such that

$$
\begin{aligned}
(\omega_y + dd^c\psi_y)^n &= e^{\eta_y}(\omega_y)^n, \\
\omega_y + dd^c\psi_y &> 0
\end{aligned}
$$

on each fiber X_y with the following normalization condition:

$$
\int_{X_y} \psi_y (\omega_y)^n = 0.
$$

Obviously, $\tilde{\rho} := \omega + dd^c\psi$ gives another fiberwise Ricci-flat metric on X. This metric is called *semi-flat* or *semi-Ricci-flat* metric on the polarized family of Calabi-Yau manifolds (cf, see [33, 37, 34]). By Remark 5.4, we have the same PDE:

$$
-\Delta c(\tilde{\rho}) = \left|\bar{\partial} v_{\tilde{\rho}}\right|_{\tilde{\rho}}^2 - \Theta_{s\bar{s}}(E).
$$

By the uniqueness of the solution of complex Monge-Ampère equation, it is easy to see that $\psi = \varphi - A(y)$ where

$$
A(y) = \int_{X_y} \varphi\omega^n.
$$

Then Theorem 1.1 and Theorem 1.2 immediately imply the following.

Corollary 7.2. *Under the hypothesis of Theorem 1.1 and Theorem 1.2, we have the following:*

(1) $p_*\tilde{\rho}^{n+1} + dd^cA$ *is positive on* Y.
(2) $\tilde{\rho} + dd^cA + K(y)\omega^{WP}$ *is positive on* X.

Remark 7.3. It is pointed by Demaiily and Eyssidieux that the fiberwise Ricci-flat metric in Theorem 1.1 and the semi-Ricci-flat metric are not uniquely determined in the cohomology class $[\omega]$. More precisely, even if ω_1 and ω_2 are Kähler metrics in X which are in the same cohomology class $[\omega]$, the fiberwise Ricci-flat metrics constructed in Theorem 1.1 (or semi-Ricci-flat metrics above) are different. Hence it is interesting to ask the canonical way to define the fiberwise Ricci-flat metric on Calabi-Yau fibrations, which is uniquely determined in each Kähler class $[\omega]$ in X.

8. An example : a family of elliptic curves

In this section, we compute the fiberwise Ricci-flat metric on the well known example which is the family of elliptic curves. The computation in this section is due to Magnusson. For the details, we refer [25, 26].

Let $\mathbb{H}$ be a upper half plane in $\mathbb{C}$. Let (z,s) be a Euclidean coordinate on $\mathbb{C}\times\mathbb{H}$. Define a group G by

$$G = \{g_{n,m} : g_{n,m}(z,s) = (z+n+ms, s).\}$$

Then G acts on $\mathbb{C}\times\mathbb{H}$ properly discontinuously. The quotient space $\mathbb{C}\times\mathbb{H}/G$ forms a universal family of elliptic curves, call it X.

The $(1,1)$-form $\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}$ on $\mathbb{C}$ descends to a Ricci-flat Kähler form on each X_s. Note that

$$\mathrm{Vol}(X_s) = \int_{X_s}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z} = \mathrm{Im}\ s.$$

Since dz is a nonvanishing holomorphic section of the direct image of the relative canonical line bundle, the curvature $\Theta(E)$ is

$$\begin{aligned}\Theta(E) &= -dd^c\log\|dz\|^2 = -dd^c\log\int_{X_s}\sqrt{-1}dz\wedge d\bar{z}\\ &= -dd^c\log\mathrm{Im}\ s = \frac{1}{|s-\bar{s}|^2}\sqrt{-1}ds\wedge d\bar{s}.\end{aligned}$$

There exists a Kähler form ρ on X such that $\pi^{-1}(\rho) = \hat{\rho}$ is written by the following:

$$\hat{\rho} = \sqrt{-1}\left(h_{s\bar{s}}ds\wedge d\bar{s} + h_{s\bar{s}}ds\wedge dz^{\bar{z}} + h_{z\bar{s}}dz\wedge d\bar{s} + h_{z\bar{z}}dz\wedge d\bar{z}\right).$$

where

$$\begin{pmatrix} h_{s\bar{s}} & h_{s\bar{z}}\\ h_{z\bar{s}} & h_{z\bar{z}}\end{pmatrix} = \begin{pmatrix} \frac{1}{(\mathrm{Im}\ s)^2} + \frac{1}{(\mathrm{Im}\ s)}\cdot\left(\frac{z-\bar{z}}{s-\bar{s}}\right)^2 & -\frac{1}{\mathrm{Im}\ s}\cdot\frac{z-\bar{z}}{s-\bar{s}}\\ -\frac{1}{\mathrm{Im}\ s}\cdot\frac{z-\bar{z}}{s-\bar{s}} & \frac{1}{\mathrm{Im}\ s}\end{pmatrix}.$$

It is easy to see that $g^*\hat{\rho} = \hat{\rho}$ for all $g\in G$. Denote by $v = \partial/\partial s$. The horizontal lift of v_ρ with respect to ρ is computed by

$$v_\rho = \frac{\partial}{\partial s} - h_{s\bar{z}}h^{\bar{z}z}\frac{\partial}{\partial z} = \frac{\partial}{\partial s} + \frac{z-\bar{z}}{s-\bar{s}}\frac{\partial}{\partial z}.$$

It follows that

$$\bar{\partial} v_\rho = -\frac{1}{s-\bar{s}} \frac{\partial}{\partial z} \otimes d\bar{z}.$$

It is easy to see that this is the harmonic representative of K_s. Hence we have

$$\left|\bar{\partial} v_\rho\right|_\rho^2 = \frac{1}{|s-\bar{s}|^2}.$$

In particular, $\left|\bar{\partial} v_\rho\right|_\rho$ is a function which depends only on s-variable. The geodesic curvature $c(\rho)$ is computed by

$$\begin{aligned} c(\rho) &= h_{s\bar{s}} - h_{s\bar{z}} h^{z\bar{z}} h_{\bar{z}s} \\ &= \frac{1}{(\mathrm{Im}\ s)^2} + \frac{1}{(\mathrm{Im}\ s)} \cdot \left(\frac{z-\bar{z}}{s-\bar{s}}\right)^2 - \left(\frac{1}{\mathrm{Im}\ s} \cdot \frac{z-\bar{z}}{s-\bar{s}}\right)^2 \cdot \mathrm{Im}\ s \\ &= \frac{1}{(\mathrm{Im}\ s)^2} > 0. \end{aligned}$$

Therefore, the fiberwise Ricci-flat metric ρ is positive on X. The direct image of ρ^2 is given as follows:

$$p_*\rho^2 = \int_{X_s} \rho^2 = \int_{X_s} c(\rho)\rho \wedge \sqrt{-1} ds \wedge d\bar{s} = \frac{1}{(\mathrm{Im}\ s)^2} \sqrt{-1} ds \wedge d\bar{s}.$$

References

[1] Aubin, T., *Equations du type Monge-Ampère sur les variétés kähleriennes compactes*, C. R. Acad. Sci. Paris Sér. A-B 283 (1976), no. 3, Aiii, A119–A121.

[2] Aubin, T., *Nonlinear analysis on manifolds. Monge-Ampère equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 252. Springer-Verlag, New York, 1982. xii+204 pp.

[3] Bando, S., Mabuchi, T, *Uniqueness of Einstein Khler metrics modulo connected group actions*, Algebraic geometry, Sendai, 1985, 1140, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam, 1987.

[4] Berndtsson, B., *Curvature of vector bundles associated to holomorphic fibrations*, Ann. of Math. (2) 169 (2009), no. 2, 531–560.

[5] Berndtsson, B., *Strict and nonstrict positivity of direct image bundles*, Math. Z. (2011), no. 3–4, 1201–1218.

[6] Berndtsson, B., Păun, M., *Bergman kernels and the pseudoeffectivity of relative canonical bundles*, Duke Math. J. 145 (2008), no. 2, 341–378

[7] Braun, M., *Positivität relativer kanonischer Bündel und Krmmung höherer direkter Bildgarben auf Familien von Calabi-Yau-Mannigfaltigkeiten*, Doctoral thesis in Philipps-Universität Marburg.

[8] Cheeger, J, Yau, S.-T., *A lower bound on heat kernel*, Comm. Pure Appl. Math. 34 (1981), no. 4, 465–480.

[9] Choi, Y.-J., *Variations of Kähler-Einstein metrics on strongly pseudoconvex domains*, Math. Ann. 362 (2015) no. 1-2, 121–146.

[10] Choi, Y.-J., *A study of variations of pesudoconex domains via Kähler-Einstein metrics*, Math. Z. 281 (2015), no. 1-2, 299–314.

[11] DeTurck, D. M., Kazdan, J. L., *Some regularity theorems in Riemannian geometry*, Ann. Sci. Ecole Norm. Sup. (4) 14 (1981), no. 3, 249–260.

[12] Di Nezza, E., Lu, H. C., *Complex Monge-Ampère equations on quasi-projective varieties*, arXiv:1401.6398.

[13] Evans, L. C., *Classical solutions of fully nonlinear, convex, second order elliptic equations*, Comm. Pure Appl. Math 25 (1982), 333363.

[14] Fujiki, A., Schumacher, G., *The moduli space of extremal compact Khler manifolds and generalized Weil-Petersson metrics*, Publ. Res. Inst. Math. Sci. 26 (1990), no. 1, 101183.

[15] Guedj, V., Zeriahi, A., *Intrinsic capacities on compact Kähler manifolds*, J. Geom. Anal. 15 (2005), no. 4, 607–639.

[16] Guenancia, H., *Families of conic Khler-Einstein metrics*, arXiv:1605.04348.

[17] Gilbarg, D., Trudinger, N, *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977. x+401 pp.

[18] Griffiths, P.A., *Curvature properties of the Hodge bundles (Notes written by Loring Tu) Topics in Transcendental Algebraic Geometry*, Annals of Mathematics Studies. Princeton University Press, Princeton (1984)

[19] Kodaira, K. *Complex manifolds and deformation of complex structures*, Grundlehren der Mathematischen Wissenschaften, 283. Springer-Verlag, New York, 1986. x+465 pp.

[20] Kodaira, K., Spencer, D. C., *On deformations of complex analytic structures. I, II*, Ann. of Math. (2) 67 1958 328–466.

[21] Kołodziej, S., *The complex Monge-Ampère equation*, Acta Math., 180 (1998), 69–117.

[22] Kołodziej, S., *The Monge-Ampère equation on compact Kähler manifolds*, Indiana Univ. Math. Jour., 52 (2003), 667–686.

[23] Krylov, N. V., *Boundedly nonhomogeneous elliptic and parabolic equations*, Izvestia Akad. Nauk. SSSR 46 (1982), 487523; English translation in Math. USSR Izv. 20 (1983), no. 3, 459–492.

[24] Jolany, H., *Log Song-Tian program along conical Kähler Ricci flow*, doctoral thesis in University of Lille1 and Princeton University, in preparation.

[25] Magnusson, G., *Métriques naturelles associées aux familles de variétés Kahlériennes compactes*, Ph. D. Thesis (2012), https://tel.archives-ouvertes.fr/tel-00849096.

[26] Magnússon, G., *A natural Hermitian metric associated with local universal families of compact Kähler manifolds with zero first Chern class*, C. R. Math. Acad. Sci. Paris 350 (2012), no. 1-2, 6366.

[27] Păun, M., *Regularity properties of the degenerate Monge-Ampère equations on compact Kähler manifolds*, Chin. Ann. Math. 29B(6), 2008, 623–630.

[28] Păun, M., *Relative adjoint transcendental classes and Albanese maps of compact Kähler manifolds with nef Ricci curvature*, arXiv:1209.2195[math.CV]

[29] Popovich, D., *Holomorphic deformations of balanced Calabi-Yau $\partial\bar{\partial}$-manifolds*, arXiv:1304.0331v1 [math.AG].

[30] Schumacher, G., *Positivity of relative canonical bundles and applications*, Invent. Math. 190 (2012), no. 1, 1–56.

[31] Semmes, S., *Interpolation of Banach spaces, differential geometry and differential equations*, Rev. Mat. Iberoamericana 4(1), 155–176 (1988).

[32] Siu, Y.-T., *Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics*, DMV Seminar, 8. Birkhuser Verlag, Basel, 1987. 171 pp.

[33] Song, J., Tian, G., *The Kahler-Ricci flow on surfaces of positive Kodaira dimension*, Invent. Math. 170 (2007) 609653.

[34] Song, J., Weinkove, B., *An introduction to the Kahler-Ricci flow, in: An introduction to the Kahler-Ricci flow* (Selected papers based on the presentations at several meetings of the ANR project MACK), Lecture No- tes in Mathematics 2086, Springer, Cham, 2013, pp. 89188.

[35] Sturm, K. T., *Heat kernel bounds on manifolds*, Math. Ann. 292, 149–162 (1992)

[36] Tian, G., *Canonical metrics in Kähler geometry*, Notes taken by Meike Akveld. Lectures in Mathematics ETH Zrich. Birkhäuser Verlag, Basel, 2000. vi+101 pp.

[37] Tosatti, V., *KAWA lecture notes on the Kähler-Ricci flow*, arXiv:1508.04823

[38] Yau, S.-T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math., no. 31(3), 339–411, 1978.

[39] Zeriahi, A., *Volume and capacity of sub level sets of a Lelong class of plurisubharmonic functions*, Indiana Univ. Math. J. 50 (2001), 671–703.

Université Grenoble-Alpes, Institut Fourier, BP74, 100 rue des maths, 38400 Saint-Martin d'Hères, France

E-mail address: `youngjun.choi@univ-grenoble-alpes.fr, choiyj0911@gmail.com`